

Approximation of analytic functions in Korobov spaces

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Abstract

We study multivariate L_2 -approximation for a weighted Korobov space of analytic periodic functions for which the Fourier coefficients decay exponentially fast. The weights are defined, in particular, in terms of two sequences $\mathbf{a} = \{a_j\}$ and $\mathbf{b} = \{b_j\}$ of numbers no less than one. Let $e^{L_2\text{-app},\Lambda}(n, s)$ be the minimal worst-case error of all algorithms that use n information functionals from the class Λ in the s -variate case. We consider two classes Λ : the class Λ^{all} consists of all linear functionals and the class Λ^{std} consists of only function evaluations.

We study (EXP) exponential convergence. This means that

$$e^{L_2\text{-app},\Lambda}(n, s) \leq C(s) q^{(n/C_1(s))^{p(s)}} \quad \text{for all } n, s \in \mathbb{N}$$

where $q \in (0, 1)$, and $C, C_1, p : \mathbb{N} \rightarrow (0, \infty)$. If we can take $p(s) = p > 0$ for all s then we speak of (UEXP) uniform exponential convergence. We also study EXP and UEXP with (WT) weak, (PT) polynomial and (SPT) strong polynomial tractability. These concepts are defined as follows. Let $n(\varepsilon, s)$ be the minimal n for which $e^{L_2\text{-app},\Lambda}(n, s) \leq \varepsilon$. Then WT holds iff $\lim_{s+\log \varepsilon^{-1} \rightarrow \infty} (\log n(\varepsilon, s)) / (s + \log \varepsilon^{-1}) = 0$, PT holds iff there are c, τ_1, τ_2 such that $n(\varepsilon, s) \leq cs^{\tau_1} (1 + \log \varepsilon^{-1})^{\tau_2}$ for all s and $\varepsilon \in (0, 1)$, and finally SPT holds iff the last estimate holds for $\tau_1 = 0$. The infimum of τ_2 for which SPT holds is called the exponent τ^* of SPT. We prove that the results are the same for both classes Λ , and:

- EXP holds for any \mathbf{a}, \mathbf{b} and ω .
- UEXP holds iff $B := \sum_{j=1}^{\infty} 1/b_j < \infty$ and the largest p is $1/B$.
- WT+EXP holds iff $\lim_j a_j = \infty$.
- WT+UEXP holds iff $B < \infty$ and $\lim_j a_j = \infty$.
- The notions of PT and SPT with EXP or UEXP are equivalent, and hold iff $B < \infty$ and $\alpha^* := \liminf_{j \rightarrow \infty} (\log a_j)/j > 0$. Then

$$\max(B, (\log 3)/\alpha^*) \leq \tau^* \leq B + (\log 3)/\alpha^*,$$

and $\tau^* = B$ for $\alpha^* = \infty$.

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1 Introduction

We study approximation of s -variate functions defined on the unit cube $[0, 1]^s$ with the worst-case error measured in the L_2 norm. Multivariate approximation is a problem that has been studied in a vast number of papers from many different perspectives. We consider analytic periodic functions belonging to a weighted Korobov space. We present necessary and sufficient conditions on the decay of the Fourier coefficients under which we can achieve exponential and uniform exponential convergence with various notions of tractability.

We approximate functions by algorithms that use n information evaluations. We either allow information evaluations from the class Λ^{all} of all continuous linear functionals or from the class Λ^{std} of standard information which consists of only function evaluations.

For large s , it is important to study how the errors of algorithms depend not only on n but also on s . The information complexity $n^{L_2-\text{app},\Lambda}(\varepsilon, s)$ is the minimal number n for which there exists an algorithm using n information evaluations from the class $\Lambda \in \{\Lambda^{\text{all}}, \Lambda^{\text{std}}\}$ with an error at most ε in the s -variate case. The information complexity is proportional to the minimal cost of computing an ε -approximation since linear algorithms are optimal and their cost is proportional to $n^{L_2-\text{app},\Lambda}(\varepsilon, s)$.

We would like to control how $n^{L_2-\text{app},\Lambda}(\varepsilon, s)$ depends on ε^{-1} and s . In the standard study of tractability, see [7, 8, 9], *weak tractability* means that $n^{L_2-\text{app},\Lambda}(\varepsilon, s)$ is *not* exponentially dependent on ε^{-1} and s . Furthermore, *polynomial tractability* means that $n^{L_2-\text{app},\Lambda}(\varepsilon, s)$ is polynomially bounded by $C s^q \varepsilon^{-p}$ for some C, q and p independent of $\varepsilon \in (0, 1)$ and $s \in \mathbb{N}$. If $q = 0$ then we have *strong polynomial tractability*.

Typically, $n^{L_2-\text{app},\Lambda}(\varepsilon, s)$ is polynomially dependent on ε^{-1} and s for weighted classes of smooth functions. The notion of weighted function classes means that the successive variables and groups of variables are moderated by certain weights. For sufficiently fast decaying weights, the information complexity depends at most polynomially on s , and we obtain *polynomial tractability*, or even *strong polynomial tractability*.

These notions of tractability are suitable for problems for which smoothness of functions is finite. This means that functions are differentiable only finitely many times. Then the minimal errors of algorithms enjoy polynomial convergence and are bounded by $C(s) n^{-\tau}$, for some positive $C(s)$ which depends only on s and some positive τ which depends on the smoothness of functions. For many classes of such functions we know the largest τ which grows with increasing smoothness and decreasing weights. Furthermore, weak tractability holds if $\log C(s) = o(s)$, whereas polynomial tractability holds if $C(s)$ is polynomially dependent on s , and strong polynomial tractability holds if $C(s)$ is uniformly bounded in s .

It seems to us that the case of analytic or infinitely many times differentiable functions is also of interest. For such classes of functions we would like to replace polynomial convergence by exponential convergence, and study the same notions of tractability in terms of $(1 + \log \varepsilon^{-1}, s)$ instead of (ε^{-1}, s) . More precisely, let $e^{L_2-\text{app},\Lambda}(n, s)$ be the minimal worst-case error among all algorithms that use n information evaluations from a permissible class Λ in the s -variate case. By exponential convergence of the n th minimal approximation error we mean that

$$e^{L_2-\text{app},\Lambda}(n, s) \leq C(s) q^{(n/C_1(s))^{p(s)}} \quad \text{for all } n, s \in \mathbb{N}.$$

Here, $q \in (0, 1)$ is independent of s , whereas C, C_1 , and p are allowed to be dependent on s .

We speak of uniform exponential convergence if p can be replaced by a positive number independent of s . A priori it is not obvious what we should require about $C(s)$, $C_1(s)$ and $p(s)$ although, clearly, the smaller $C(s)$ and $C_1(s)$ the better, and we would like to have $p(s)$ as large as possible. Obviously, if we do not care about the dependence on s then the mere existence of $C(s)$, $C_1(s)$ and $p(s)$ is enough.

The last bound on $e^{L_2\text{-app},\Lambda}(n, s)$ yields

$$n^{L_2\text{-app},\Lambda}(\varepsilon, s) \leq \left\lceil C_1(s) \left(\frac{\log C(s) + \log \varepsilon^{-1}}{\log q^{-1}} \right)^{1/p(s)} \right\rceil \quad \text{for all } s \in \mathbb{N} \text{ and } \varepsilon \in (0, 1).$$

Exponential convergence implies that asymptotically with respect to ε tending to zero, we need $\mathcal{O}(\log^{1/p(s)} \varepsilon^{-1})$ information evaluations to compute an ε -approximation to functions from the Korobov space. (Throughout the paper \log means the natural logarithm and $\log^r x$ means $[\log x]^r$.)

Tractability with exponential or uniform exponential convergence means that we would like to replace ε^{-1} by $1 + \log \varepsilon^{-1}$ and guarantee the same properties on $n^{L_2\text{-app},\Lambda}(\varepsilon, s)$ as for the standard case. This means that (WT) weak tractability holds iff

$$\lim_{s + \log \varepsilon^{-1} \rightarrow \infty} \frac{\log n^{L_2\text{-app},\Lambda}(\varepsilon, s)}{s + \log \varepsilon^{-1}} = 0,$$

whereas (PT) polynomial tractability holds iff there are non-negative numbers c, τ_1, τ_2 such that

$$n^{L_2\text{-app},\Lambda}(\varepsilon, s) \leq c s^{\tau_1} (1 + \log \varepsilon^{-1})^{\tau_2} \quad \text{for all } s \in \mathbb{N}, \varepsilon \in (0, 1).$$

If $\tau_1 = 0$ in the last bound we speak of (SPT) strong polynomial tractability, and then τ^* being the infimum of τ_2 is called the exponent of SPT.

For instance, uniform exponential convergence implies weak tractability if

$$C(s) = \exp(\exp(o(s))) \quad \text{and} \quad C_1(s) = \exp(o(s)) \quad \text{as } s \rightarrow \infty.$$

These conditions are rather weak since $C(s)$ can be almost doubly exponential and $C_1(s)$ almost exponential in s .

Furthermore, uniform exponential convergence implies polynomial tractability if for some non-negative η_1 and η_2 we have

$$C(s) = \exp(\mathcal{O}(s^{\eta_1})) \quad \text{and} \quad C_1(s) = \mathcal{O}(s^{\eta_2}) \quad \text{as } s \rightarrow \infty.$$

If $\eta_1 = \eta_2 = 0$ then we have strong polynomial tractability.

Uniform exponential convergence with weak, polynomial and strong polynomial tractability was studied in the papers [2] and [4] for multivariate integration in weighted Korobov spaces with exponentially fast decaying Fourier coefficients. However, the notion of weak tractability was defined differently in a more demanding way, see Section 9 for more details. In the current paper, we deal with multivariate approximation in the worst-case setting for the same class of functions. We study exponential and uniform exponential convergence and various notions of tractability defined as above.

We find it interesting that all results presented in this paper are exactly the same for both classes Λ^{all} and Λ^{std} . This is surprising since the class Λ^{std} is much smaller than

the class Λ^{all} . This is very good news since usually in the computational practice we can only use function values, i.e., the class Λ^{std} . Furthermore, all our results are constructive or semi-constructive¹. That is, we provide algorithms that use only function values and for which we achieve exponential and uniform exponential convergence with WT, PT or SPT. The sample points used by these algorithms are from regular grids with varying mesh-sizes for successive variables. Such grids were also successfully used for multivariate integration in the previous papers [2] and [4].

For the Korobov class of functions f considered here, the decay of the Fourier coefficients $\widehat{f}(\mathbf{h})$ is defined by two sequences $\mathbf{a} = \{a_j\}$ and $\mathbf{b} = \{b_j\}$, and by a parameter $\omega \in (0, 1)$. Here \mathbf{a} and \mathbf{b} are two sequences of real numbers bounded below by 1, see Section 2 for further details. We assume that

$$\sum_{\mathbf{h} \in \mathbb{Z}^s} |\widehat{f}(\mathbf{h})|^2 \omega_{\mathbf{h}}^{-1} < \infty,$$

where

$$\omega_{\mathbf{h}} = \omega^{\sum_{j=1}^s a_j |h_j|^{b_j}} \quad \text{for all} \quad \mathbf{h} = (h_1, h_2, \dots, h_s) \in \mathbb{Z}^s.$$

We study for which $(\mathbf{a}, \mathbf{b}, \omega)$ we have exponential and uniform exponential convergence without or with various notions of tractability. It turns out that ω only effects the factors in our estimates. These factors go to infinity as ω tends to one.

We are going to show that exponential convergence holds for any choice of \mathbf{a} and \mathbf{b} , whereas uniform exponential convergence holds iff

$$B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty,$$

independently of \mathbf{a} . Furthermore, the largest rate $p(s)$ for exponential convergence is $1/B(s)$, where

$$B(s) = \sum_{j=1}^s \frac{1}{b_j},$$

and for uniform exponential convergence the largest rate p is $1/B$.

We prove that (WT+EXP) weak tractability with exponential convergence holds iff

$$\lim_{j \rightarrow \infty} a_j = \infty,$$

and (WT+UEXP) weak tractability with uniform exponential convergence holds iff

$$B < \infty \quad \text{and} \quad \lim_{j \rightarrow \infty} a_j = \infty.$$

The notions of polynomial and strong polynomial tractability with exponential or uniform exponential convergence are equivalent. Furthermore, the strongest notion of tractability, namely strong polynomial tractability with uniform exponential convergence, holds iff

$$B < \infty \quad \text{and} \quad \alpha^* = \liminf_{j \rightarrow \infty} \frac{\log a_j}{j} > 0,$$

¹Semi-construction is only used for the class Λ^{std} when we want to achieve WT with UEXP, see Section 8.4.

and then the exponent τ^* of SPT satisfies

$$\max\left(B, \frac{\log 3}{\alpha^*}\right) \leq \tau^* \leq B + \frac{\log 3}{\alpha^*}.$$

We comment on the assumption that $\alpha^* > 0$. This means that the a_j are exponentially large in j for large j . Indeed, $\alpha^* > 0$ implies that for any $\delta \in (0, \alpha^*)$ there is j_δ^* such that

$$a_j \geq \exp(\delta j) \quad \text{for all } j \geq j_\delta^*. \quad (1)$$

Obviously, it may happen that $\alpha^* = \infty$. Then we know the exponent of SPT exactly,

$$\tau^* = B.$$

Note that this happens if, for instance, $a_j \geq \exp(\alpha b_j)$ for large j and for some $\alpha > 0$. Indeed, then

$$\alpha^* \geq \liminf_{j \rightarrow \infty} \frac{\alpha b_j}{j} = \infty,$$

since $B < \infty$ implies that $\liminf_{j \rightarrow \infty} b_j/j = \infty$.

The rest of the paper is structured as follows. We give detailed information on the Korobov space in Section 2, and on L_2 -approximation and tractability in Section 3. Our main results are summarized in Section 4. The proofs for the class Λ^{all} are in Section 6 using preliminary observations from Section 5. The proofs for the class Λ^{std} are in Section 8 using preliminary observations from Section 7. In Section 9 we compare the approximation problem considered in this paper with the integration problem considered in [4]. Analyticity of functions from the Korobov space considered in this paper is shown in Section 10.

2 The Korobov space $H(K_{s,\mathbf{a},\mathbf{b}})$

The Korobov space $H(K_{s,\mathbf{a},\mathbf{b}})$ discussed in this section is a Hilbert space with a reproducing kernel. For general information on reproducing kernel Hilbert spaces we refer to [1].

Let $\mathbf{a} = \{a_j\}_{j \geq 1}$ and $\mathbf{b} = \{b_j\}_{j \geq 1}$ be two sequences of real positive weights such that

$$b_j \geq 1 \quad \text{and} \quad a_j \geq 1 \quad \text{for all } j = 1, 2, \dots \quad (2)$$

Throughout the paper we assume, without loss of generality, that

$$a_1 \leq a_2 \leq a_3 \leq \dots$$

Fix $\omega \in (0, 1)$. Denote

$$\omega_{\mathbf{h}} = \omega^{\sum_{j=1}^s a_j |h_j|^{b_j}} \quad \text{for all } \mathbf{h} = (h_1, h_2, \dots, h_s) \in \mathbb{Z}^s.$$

We consider a Korobov space of complex-valued one-periodic functions defined on $[0, 1]^s$ with a reproducing kernel of the form

$$K_{s,\mathbf{a},\mathbf{b}}(\mathbf{x}, \mathbf{y}) = \sum_{\mathbf{h} \in \mathbb{Z}^s} \omega_{\mathbf{h}} \exp(2\pi i \mathbf{h} \cdot (\mathbf{x} - \mathbf{y})) \quad \text{for all } \mathbf{x}, \mathbf{y} \in [0, 1]^s,$$

with the usual dot product

$$\mathbf{h} \cdot (\mathbf{x} - \mathbf{y}) = \sum_{j=1}^s h_j(x_j - y_j),$$

where h_j, x_j, y_j are the j th components of the vectors $\mathbf{h}, \mathbf{x}, \mathbf{y}$, respectively, and $\mathbf{i} = \sqrt{-1}$.

The kernel $K_{s,\mathbf{a},\mathbf{b}}$ is well defined since

$$|K_{s,\mathbf{a},\mathbf{b}}(\mathbf{x}, \mathbf{y})| \leq K_{s,\mathbf{a},\mathbf{b}}(\mathbf{x}, \mathbf{x}) = \prod_{j=1}^s \left(1 + 2 \sum_{h=1}^{\infty} \omega^{a_j h^{b_j}} \right) < \infty. \quad (3)$$

The last series is indeed finite since

$$\sum_{h=1}^{\infty} \omega^{a_j h^{b_j}} \leq \sum_{h=1}^{\infty} \omega^h = \frac{\omega}{1 - \omega} < \infty.$$

The Korobov space with reproducing kernel $K_{s,\mathbf{a},\mathbf{b}}$ is a reproducing kernel Hilbert space and is denoted by $H(K_{s,\mathbf{a},\mathbf{b}})$. We suppress the dependence on ω in the notation since ω will be fixed throughout the paper and \mathbf{a} and \mathbf{b} will be varied.

Clearly, functions from $H(K_{s,\mathbf{a},\mathbf{b}})$ are infinitely many times differentiable, see [2]. They are also analytic as shown in Section 10.

For $f \in H(K_{s,\mathbf{a},\mathbf{b}})$ we have

$$f(\mathbf{x}) = \sum_{\mathbf{h} \in \mathbb{Z}^s} \widehat{f}(\mathbf{h}) \exp(2\pi \mathbf{i} \mathbf{h} \cdot \mathbf{x}) \quad \text{for all } \mathbf{x} \in [0, 1]^s,$$

where $\widehat{f}(\mathbf{h}) = \int_{[0,1]^s} f(\mathbf{x}) \exp(-2\pi \mathbf{i} \mathbf{h} \cdot \mathbf{x}) d\mathbf{x}$ is the \mathbf{h} th Fourier coefficient. The inner product of f and g from $H(K_{s,\mathbf{a},\mathbf{b}})$ is given by

$$\langle f, g \rangle_{H(K_{s,\mathbf{a},\mathbf{b}})} = \sum_{\mathbf{h} \in \mathbb{Z}^s} \widehat{f}(\mathbf{h}) \overline{\widehat{g}(\mathbf{h})} \omega_{\mathbf{h}}^{-1}$$

and the norm of f from $H(K_{s,\mathbf{a},\mathbf{b}})$ by

$$\|f\|_{H(K_{s,\mathbf{a},\mathbf{b}})} = \left(\sum_{\mathbf{h} \in \mathbb{Z}^s} |\widehat{f}(\mathbf{h})|^2 \omega_{\mathbf{h}}^{-1} \right)^{1/2} < \infty.$$

Define the functions

$$e_{\mathbf{h}}(\mathbf{x}) = \exp(2\pi \mathbf{i} \mathbf{h} \cdot \mathbf{x}) \omega_{\mathbf{h}}^{1/2} \quad \text{for all } \mathbf{x} \in [0, 1]^s. \quad (4)$$

Then $\{e_{\mathbf{h}}\}_{\mathbf{h} \in \mathbb{Z}^s}$ is a complete orthonormal basis of the Korobov space $H(K_{s,\mathbf{a},\mathbf{b}})$.

Integration of functions from $H(K_{s,\mathbf{a},\mathbf{b}})$ was already considered in [4] and, in the case $a_j = b_j = 1$ for all $j \in \mathbb{N}$, also in [2]. In this paper we consider the problem of multivariate approximation in the L_2 norm which we shortly call L_2 -approximation.

3 L_2 -approximation

In this section we consider L_2 -approximation of functions from $H(K_{s,\mathbf{a},\mathbf{b}})$. This problem is defined as an approximation of the embedding from the Korobov space $H(K_{s,\mathbf{a},\mathbf{b}})$ to the space $L_2([0,1]^s)$, i.e.,

$$\text{EMB}_s : H(K_{s,\mathbf{a},\mathbf{b}}) \rightarrow L_2([0,1]^s) \quad \text{given by} \quad \text{EMB}_s(f) = f.$$

Without loss of generality, see, e.g., [10], we approximate EMB_s by linear algorithms $A_{n,s}$ of the form

$$A_{n,s}(f) = \sum_{k=1}^n \alpha_k L_k(f) \quad \text{for} \quad f \in H(K_{s,\mathbf{a},\mathbf{b}}), \quad (5)$$

where each α_k is a function from $L_2([0,1]^s)$ and each L_k is a continuous linear functional defined on $H(K_{s,\mathbf{a},\mathbf{b}})$ from a permissible class Λ of information. We consider two classes:

- $\Lambda = \Lambda^{\text{all}}$, the class of all continuous linear functionals defined on $H(K_{s,\mathbf{a},\mathbf{b}})$. Since $H(K_{s,\mathbf{a},\mathbf{b}})$ is a Hilbert space then for every $L_k \in \Lambda^{\text{all}}$ there exists a function f_k from $H(K_{s,\mathbf{a},\mathbf{b}})$ such that $L_k(f) = \langle f, f_k \rangle_{H(K_{s,\mathbf{a},\mathbf{b}})}$ for all $f \in H(K_{s,\mathbf{a},\mathbf{b}})$.
- $\Lambda = \Lambda^{\text{std}}$, the class of standard information consisting only of function evaluations. That is, $L_k \in \Lambda^{\text{std}}$ iff there exists $\mathbf{x}_k \in [0,1]^s$ such that $L_k(f) = f(\mathbf{x}_k)$ for all $f \in H(K_{s,\mathbf{a},\mathbf{b}})$.

Since $H(K_{s,\mathbf{a},\mathbf{b}})$ is a reproducing kernel Hilbert space, function evaluations are continuous linear functionals and therefore $\Lambda^{\text{std}} \subseteq \Lambda^{\text{all}}$. More precisely,

$$L_k(f) = f(\mathbf{x}_k) = \langle f, K_{s,\mathbf{a},\mathbf{b}}(\cdot, \mathbf{x}_k) \rangle_{H(K_{s,\mathbf{a},\mathbf{b}})} \quad \text{and} \quad \|L_k\| = \|K_{s,\mathbf{a},\mathbf{b}}\|_{H(K_{s,\mathbf{a},\mathbf{b}})} = K_{s,\mathbf{a},\mathbf{b}}^{1/2}(\mathbf{x}_k, \mathbf{x}_k).$$

The *worst-case error* of the algorithm $A_{n,s}$ is defined as

$$e^{L_2\text{-app}}(H(K_{s,\mathbf{a},\mathbf{b}}), A_{n,s}) := \sup_{\substack{f \in H(K_{s,\mathbf{a},\mathbf{b}}) \\ \|f\|_{H(K_{s,\mathbf{a},\mathbf{b}})} \leq 1}} \|f - A_{n,s}(f)\|_{L_2([0,1]^s)}.$$

Let $e^{L_2\text{-app},\Lambda}(n, s)$ be the n th minimal worst-case error,

$$e^{L_2\text{-app},\Lambda}(n, s) = \inf_{A_{n,s}} e^{L_2\text{-app}}(H(K_{s,\mathbf{a},\mathbf{b}}), A_{n,s}),$$

where the infimum is taken over all linear algorithms $A_{n,s}$ using information from the class Λ . For $n = 0$ we simply approximate f by zero, and the initial error is

$$e^{L_2\text{-app},\Lambda}(0, s) = \|\text{EMB}_s\| = \sup_{\substack{f \in H(K_{s,\mathbf{a},\mathbf{b}}) \\ \|f\|_{H(K_{s,\mathbf{a},\mathbf{b}})} \leq 1}} \|f\|_{L_2([0,1]^s)} = 1.$$

This means that L_2 -approximation is well normalized for all $s \in \mathbb{N}$.

We study exponential convergence in this paper. Suppose first that $s \in \mathbb{N}$ is fixed. Then we hope that everyone would agree that exponential convergence for $e^{L_2\text{-app},\Lambda}(n, s)$ means that there exist functions $q : \mathbb{N} \rightarrow (0, 1)$ and $p, C : \mathbb{N} \rightarrow (0, \infty)$ such that

$$e^{L_2\text{-app},\Lambda}(n, s) \leq C(s) q(s) n^{p(s)} \quad \text{for all} \quad n \in \mathbb{N}.$$

Obviously, the functions $q(\cdot)$ and $p(\cdot)$ are not uniquely defined. For instance, we can take an arbitrary number $q \in (0, 1)$, define the function C_1 as

$$C_1(s) = \left(\frac{\log q}{\log q(s)} \right)^{1/p(s)}$$

and then

$$C(s) q(s)^{n^{p(s)}} = C(s) q^{(n/C_1(s))^{p(s)}}.$$

We prefer to work with the latter bound which was already considered in [4] for multivariate integration.

We say that we achieve *exponential convergence* for $e^{L_2-\text{app},\Lambda}(n, s)$ if there exist a number $q \in (0, 1)$ and functions $p, C, C_1 : \mathbb{N} \rightarrow (0, \infty)$ such that

$$e^{L_2-\text{app},\Lambda}(n, s) \leq C(s) q^{(n/C_1(s))^{p(s)}} \quad \text{for all } n \in \mathbb{N}. \quad (6)$$

If (6) holds we would like to find the largest possible rate $p(s)$ of exponential convergence defined as

$$p^*(s) = \sup\{p(s) : p(s) \text{ satisfies (6)}\}. \quad (7)$$

We say that we achieve *uniform exponential convergence* for $e^{L_2-\text{app},\Lambda}(n, s)$ if the function p in (6) can be taken as a constant function, i.e., $p(s) = p > 0$ for all $s \in \mathbb{N}$. Similarly, let

$$p^* = \sup\{p : p(s) = p > 0 \text{ satisfies (6) for all } s \in \mathbb{N}\}$$

denote the largest rate of uniform exponential convergence.

For $\varepsilon \in (0, 1)$, $s \in \mathbb{N}$, and $\Lambda \in \{\Lambda^{\text{all}}, \Lambda^{\text{std}}\}$, the *information complexity* is defined as

$$n^{L_2-\text{app},\Lambda}(\varepsilon, s) := \min \{n : e^{L_2-\text{app},\Lambda}(n, s) \leq \varepsilon\}.$$

Hence, $n^{L_2-\text{app},\Lambda}(\varepsilon, s)$ is the minimal number of information evaluations from Λ which is required to reduce the initial error $e_{0,s}^{L_2-\text{app}}$, which is one in our case, by a factor of $\varepsilon \in (0, 1)$. Clearly

$$n^{L_2-\text{app},\Lambda^{\text{std}}}(\varepsilon, s) \geq n^{L_2-\text{app},\Lambda^{\text{all}}}(\varepsilon, s).$$

We are ready to define tractability concepts similarly as in [2] and [4]. We stress again that these concepts correspond to the standard concepts of tractability with ε^{-1} replaced by $1 + \log \varepsilon^{-1}$. We say that we have:

- *Weak Tractability (WT)* if

$$\lim_{s + \log \varepsilon^{-1} \rightarrow \infty} \frac{\log n^{L_2-\text{app},\Lambda}(\varepsilon, s)}{s + \log \varepsilon^{-1}} = 0.$$

Here we set $\log 0 = 0$ by convention.

- *Polynomial Tractability (PT)* if there exist non-negative numbers c, τ_1, τ_2 such that

$$n^{L_2-\text{app},\Lambda}(\varepsilon, s) \leq c s^{\tau_1} (1 + \log \varepsilon^{-1})^{\tau_2} \quad \text{for all } s \in \mathbb{N}, \varepsilon \in (0, 1).$$

- *Strong Polynomial Tractability (SPT)* if there exist non-negative numbers c and τ such that

$$n^{L_2\text{-app},\Lambda}(\varepsilon, s) \leq c(1 + \log \varepsilon^{-1})^\tau \quad \text{for all } s \in \mathbb{N}, \varepsilon \in (0, 1).$$

The exponent τ^* of strong polynomial tractability is defined as the infimum of τ for which strong polynomial tractability holds.

A few comments of these notions are in order. As in [2], we note that if (6) holds then

$$n^{L_2\text{-app},\Lambda}(\varepsilon, s) \leq \left\lceil C_1(s) \left(\frac{\log C(s) + \log \varepsilon^{-1}}{\log q^{-1}} \right)^{1/p(s)} \right\rceil \quad \text{for all } s \in \mathbb{N} \text{ and } \varepsilon \in (0, 1). \quad (8)$$

Furthermore, if (8) holds then

$$e^{L_2\text{-app},\Lambda}(n+1, s) \leq C(s) q^{(n/C_1(s))^{p(s)}} \quad \text{for all } s, n \in \mathbb{N}.$$

This means that (6) and (8) are practically equivalent. Note that $1/p(s)$ determines the power of $\log \varepsilon^{-1}$ in the information complexity, whereas $\log q^{-1}$ effects only the multiplier of $\log^{1/p(s)} \varepsilon^{-1}$. From this point of view, $p(s)$ is more important than q . That is why we would like to have (6) with the largest possible $p(s)$. We shall see how to find such $p(s)$ for the parameters $(\mathbf{a}, \mathbf{b}, \omega)$ of the weighted Korobov space.

Exponential convergence implies that asymptotically, with respect to ε tending to zero, we need $\mathcal{O}(\log^{1/p(s)} \varepsilon^{-1})$ information evaluations to compute an ε -approximation to functions from the Korobov space. However, it is not clear how long we have to wait to see this nice asymptotic behavior especially for large s . This, of course, depends on how $C(s)$, $C_1(s)$ and $p(s)$ depend on s . This is the subject of tractability which is extensively studied in many papers. So far tractability has been studied in terms of s and ε^{-1} . The current state of the art on tractability can be found in [7, 8, 9]. In this paper we follow the approach of [2] and [4] and we study tractability in terms of s and $1 + \log \varepsilon^{-1}$. In particular, weak tractability means that we rule out the cases for which $n^{L_2\text{-app},\Lambda}(\varepsilon, s)$ depends exponentially on s and $\log \varepsilon^{-1}$.

For instance, assume that (6) holds. Then uniform exponential convergence implies weak tractability if

$$C(s) = \exp(\exp(o(s))) \quad \text{and} \quad C_1(s) = \exp(o(s)) \quad \text{as } s \rightarrow \infty.$$

These conditions are rather weak since $C(s)$ can be almost doubly exponential and $C_1(s)$ almost exponential in s . The definition of polynomial (and strong polynomial) tractability implies that we have uniform exponential convergence with $C(s) = e$ (where e denotes $\exp(1)$), $q = 1/e$, $C_1(s) = c s^{\tau_1}$ and $p = 1/\tau_2$. For strong polynomial tractability $C_1(s) = c$ and $\tau^* \leq 1/p^*$.

If (8) holds then we have polynomial tractability if $p := \inf_s p(s) > 0$ and there exist non-negative numbers A , A_1 and η, η_1 such that

$$C(s) \leq \exp(As^\eta) \quad \text{and} \quad C_1(s) \leq A_1 s^{\eta_1} \quad \text{for all } s \in \mathbb{N}.$$

The condition on $C(s)$ seems to be quite weak since even for singly exponential $C(s)$ we have polynomial tractability. Then $\tau_1 = \eta_1 + \eta/p$ and $\tau_2 = 1/p$. Strong polynomial tractability holds if $C(s)$ and $C_1(s)$ are uniformly bounded in s , and then $\tau^* \leq 1/p$.

4 The main results

We first present the main results of this paper. We will be using the following notational abbreviations

$$\begin{array}{ccccc} \text{EXP} & \text{UEXP} & \text{WT} & \text{PT} & \text{SPT} \\ \text{WT+EXP} & & \text{PT+EXP} & & \text{SPT+EXP} \\ \text{WT+UEXP} & & \text{PT+UEXP} & & \text{SPT+UEXP} \end{array}$$

to denote exponential and uniform exponential convergence, weak, polynomial and strong polynomial tractability, as well as weak, polynomial and strong polynomial tractability with exponential or uniform exponential convergence. We want to find relations between these concepts as well as necessary and sufficient conditions on \mathbf{a} and \mathbf{b} for which these concepts hold. As we shall see, many of these concepts are equivalent.

Theorem 1 *Consider L_2 -approximation defined over the Korobov space with kernel $K_{s,\mathbf{a},\mathbf{b}}$ with arbitrary sequences \mathbf{a} and \mathbf{b} satisfying (2). The following results hold for both classes Λ^{all} and Λ^{std} .*

1 EXP holds for arbitrary \mathbf{a} and \mathbf{b} and

$$p^*(s) = 1/B(s) \quad \text{with} \quad B(s) := \sum_{j=1}^s \frac{1}{b_j}.$$

This implies that

$$\text{WT} \Leftrightarrow \text{WT+EXP}, \quad \text{PT} \Leftrightarrow \text{PT+EXP}, \quad \text{SPT} \Leftrightarrow \text{SPT+EXP}.$$

2 UEXP holds iff \mathbf{a} is an arbitrary sequence and \mathbf{b} such that

$$B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty.$$

If so then $p^* = 1/B$ and

$$\text{WT} \Leftrightarrow \text{WT+UEXP}, \quad \text{PT} \Leftrightarrow \text{PT+UEXP}, \quad \text{SPT} \Leftrightarrow \text{SPT+UEXP}.$$

3 Polynomial (and, of course, strong polynomial) tractability implies uniform exponential convergence, $\text{PT} \Rightarrow \text{UEXP}$, i.e.,

$$\text{PT} \Leftrightarrow \text{PT+UEXP}, \quad \text{SPT} \Leftrightarrow \text{SPT+UEXP}.$$

4 We have

$$\begin{aligned} \text{WT} &\Leftrightarrow \lim_{j \rightarrow \infty} a_j = \infty, \\ \text{WT+UEXP} &\Leftrightarrow B < \infty \quad \text{and} \quad \lim_{j \rightarrow \infty} a_j = \infty. \end{aligned}$$

5 The following notions are equivalent:

$$\text{PT} \Leftrightarrow \text{PT+EXP} \Leftrightarrow \text{PT+UEXP} \Leftrightarrow \text{SPT} \Leftrightarrow \text{SPT+EXP} \Leftrightarrow \text{SPT+UEXP}.$$

6 *SPT+UEXP holds iff b_j^{-1} 's are summable and a_j 's are exponentially large in j , i.e.,*

$$B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty \quad \text{and} \quad \alpha^* := \liminf_{j \rightarrow \infty} \frac{\log a_j}{j} > 0.$$

Then the exponent τ^ of SPT satisfies*

$$\max \left(B, \frac{\log 3}{\alpha^*} \right) \leq \tau^* \leq B + \frac{\log 3}{\alpha^*}.$$

In particular, if $\alpha^ = \infty$ then $\tau^* = B$.*

We comment on Theorem 1. We already expressed our surprise in the introduction that the results are the same for both classes Λ^{std} and Λ^{all} , although the class Λ^{std} is much smaller than the class Λ^{all} . However, the proofs for both classes are different. We also stress that the results are constructive. The corresponding algorithms can be found in Section 5 for the class Λ^{all} and in Section 8 for the class Λ^{std} .

Point 1 tells us that we always have exponential convergence and the best rate is $p^*(s) = 1/B(s)$. Note that $p^*(s)$ decays with s , and if $B(s)$ goes to infinity then the rate decays to zero. The smallest rate is for $b_j = 1$ for all $j \geq 1$, for which $p^*(s) = 1/s$. Clearly, all tractability notions with or without exponential convergence are trivially equivalent.

Point 2 addresses uniform exponential convergence which holds iff b_j^{-1} 's are summable, i.e., when $B < \infty$. Then the best rate of uniform exponential convergence is $p^* = 1/B$. Obviously, for large B this rate is poor. We stress that uniform exponential convergence holds independently of \mathbf{a} . Similarly as before, as long as $B < \infty$, tractability notions with or without uniform exponential convergence are trivially equivalent.

Point 3 states that (strong) polynomial tractability implies uniform exponential convergence, i.e., $B < \infty$. This means that the notion of polynomial tractability is stronger than the notion of uniform convergence.

Point 4 addresses weak tractability which holds iff a_j 's tend to infinity. We stress that this holds independently of \mathbf{b} and independently of the rate of convergence of \mathbf{a} to infinity. We have weak tractability with uniform convergence if additionally $B < \infty$. Hence for $\lim_j a_j = \infty$ and $B = \infty$, weak tractability holds without uniform exponential convergence.

Point 5 states that, in particular, the notions of polynomial tractability and strong polynomial tractability with uniform exponential convergence are equivalent.

Point 6 presents necessary and sufficient conditions on strong polynomial tractability with uniform exponential convergence. We must assume that $B < \infty$ and $\alpha^* > 0$. The last condition means that a_j 's are exponentially large in j for large j . We only know bounds of the exponent τ^* of strong polynomial tractability. Note that for large B or small α^* the exponent τ^* is large. On the other hand, τ^* is not large if B is not large and α^* is not small. We stress that B can be sufficiently small if all b_j are sufficiently large, whereas α^* can be sufficiently large if a_j are large enough. In fact, we may even have $\alpha^* = \infty$. This holds if a_j goes to infinity faster than C^j for any $C > 1$. We already noticed in the introduction that this holds, for example, if $a_j \geq \exp(\delta b_j)$ for large j and for some $\delta > 0$. For $\alpha^* = \infty$ we know the exponent of SPT exactly,

$$\tau^* = B.$$

5 Preliminaries for the class Λ^{all}

The information complexity is known for the class Λ^{all} , see, e.g., [10, Chapter 4, Section 5.8]). It depends on the eigenpairs of the operator

$$W_s = \text{EMB}_s^* \text{EMB}_s : H(K_{s,\mathbf{a},\mathbf{b}}) \rightarrow H(K_{s,\mathbf{a},\mathbf{b}}),$$

which in our case is given by

$$W_s f = \sum_{\mathbf{h} \in \mathbb{Z}^s} \omega_{\mathbf{h}} \langle f, e_{\mathbf{h}} \rangle_{H(K_{s,\mathbf{a},\mathbf{b}})} e_{\mathbf{h}}$$

with $e_{\mathbf{h}}$ given by (4). Hence, the eigenpairs of W_s are $(\omega_{\mathbf{h}}, e_{\mathbf{h}})$ since

$$W_s e_{\mathbf{h}} = \omega_{\mathbf{h}} e_{\mathbf{h}} = \omega^{\sum_{j=1}^s a_j |h_j|^{b_j}} e_{\mathbf{h}} \quad \text{for all } \mathbf{h} \in \mathbb{Z}^s.$$

It is known that the information complexity is the number of the eigenvalues $\omega_{\mathbf{h}}$ of the operator W_s which are greater than ε^2 . More precisely, for a real M define the set

$$\begin{aligned} \mathcal{A}(s, M) &:= \{ \mathbf{h} \in \mathbb{Z}^s : \omega_{\mathbf{h}}^{-1} < M \} \\ &= \left\{ \mathbf{h} \in \mathbb{Z}^s : \omega^{-\sum_{j=1}^s a_j |h_j|^{b_j}} < M \right\}. \end{aligned} \quad (9)$$

Then

$$n^{L_2\text{-app}, \Lambda^{\text{all}}}(\varepsilon, s) = |\mathcal{A}(s, \varepsilon^{-2})|. \quad (10)$$

Furthermore, the optimal algorithm in the class Λ^{all} is the truncated Fourier series

$$A_{n,s}^{(\text{opt})}(f)(\mathbf{x}) := \sum_{\mathbf{h} \in \mathcal{A}(s, \varepsilon^{-2})} \langle f, e_{\mathbf{h}} \rangle_{H(K_{s,\mathbf{a},\mathbf{b}})} e_{\mathbf{h}} = \sum_{\mathbf{h} \in \mathcal{A}(s, \varepsilon^{-2})} \widehat{f}(\mathbf{h}) \exp(2\pi i \mathbf{h} \cdot \mathbf{x}),$$

where $n = |\mathcal{A}(s, \varepsilon^{-2})|$, which ensures that the worst-case error satisfies

$$e^{L_2\text{-app}}(H(K_{s,\mathbf{a},\mathbf{b}}), A_{n,s}^{(\text{opt})}) \leq \varepsilon.$$

For the proof of Theorem 1 and also for the further considerations in this paper we need a few properties of the set $\mathcal{A}(s, M)$ and its cardinality. Clearly, $\mathcal{A}(s, M) = \emptyset$ for all $M \leq 1$. For $\varepsilon \in (0, 1)$, let

$$x = x(\varepsilon) := \frac{\log \varepsilon^{-2}}{\log \omega^{-1}} > 0,$$

and

$$n(x, s) := \left| \left\{ \mathbf{h} \in \mathbb{Z}^s : \sum_{j=1}^s a_j |h_j|^{b_j} < x \right\} \right|.$$

Then

$$n^{L_2\text{-app}, \Lambda^{\text{all}}}(\varepsilon, s) = |\mathcal{A}(s, \varepsilon^{-2})| = n(x, s).$$

We have $n(x, s) = 1$ for all $x \in (0, a_1]$ and

$$n(x, 1) = 2 \lceil (x/a_1)^{1/b_1} \rceil - 1,$$

$$n(x, s) = n(x, s-1) + 2 \sum_{h=1}^{\lceil (x/a_s)^{1/b_s} \rceil - 1} n(x - a_s h^{b_s}, s-1).$$

Clearly, $n(y, s) \geq n(x, s) \geq n(x, s-1)$ for all $y \geq x > 0$ and $s \geq 2$. Note that for $x \leq a_s$, the last sum in $n(x, s)$ is zero and $n(x, s) = n(x, s-1)$. For $x > a_1$, define

$$j(x) = \sup\{j \in \mathbb{N} : x > a_j\}.$$

For $\lim_j a_j < \infty$ we have $j(x) = \infty$ for large x . For $\lim_j a_j = \infty$, we can replace the supremum in $j(x)$ by the maximum, and $j(x)$ is finite for all x . However, $j(x)$ tends to infinity with x .

If $j(x)$ is finite then

$$n(x, s) = n(x, j(x)) \quad \text{for all } s \geq j(x),$$

and therefore, if $j(x) < \infty$ then

$$\lim_{s \rightarrow \infty} \frac{\log n(x, s)}{s + x} = 0.$$

We now prove the following lemma.

Lemma 1

- For $x > a_1 + a_2 + \dots + a_s$ we have

$$n(x, s) \geq 3^s.$$

- For $x > a_1$ and for arbitrary $\alpha_j \in [0, 1]$ we have

$$\begin{aligned} n(x, s) &\geq \prod_{j=1}^{\min(s, j(x))} \left(2 \left\lceil \left(\frac{x}{a_j} (1 - \alpha_j) \prod_{k=j+1}^s \alpha_k \right)^{1/b_j} \right\rceil - 1 \right), \\ n(x, s) &\leq \prod_{j=1}^{\min(s, j(x))} \left(2 \left\lceil \left(\frac{x}{a_j} \right)^{1/b_j} \right\rceil - 1 \right), \end{aligned}$$

where the empty product is defined to be 1.

- For $x > a_1$ we have

$$\prod_{j=1}^s \left(2 \left\lceil \left(\frac{x}{a_j} \right)^{1/b_j} \right\rceil - 1 \right) \leq n(x, s) \leq \prod_{j=1}^{\min(s, j(x))} \left(2 \left\lceil \left(\frac{x}{a_j} \right)^{1/b_j} \right\rceil - 1 \right).$$

Proof. To prove the first point, let $A_s = \{\mathbf{h} \in \mathbb{Z}^s : h_j \in \{-1, 0, 1\}\}$. For $\mathbf{h} \in A_s$ we have

$$\sum_{j=1}^s a_j |h_j|^{b_j} \leq \sum_{j=1}^s a_j < x.$$

Hence $3^s = |A_s| \leq n(x, s)$, as claimed.

We turn to the second point. It is easier to prove the upper bound on $n(x, s)$. From the recurrence relation on $n(x, s)$ we have

$$n(x, s) \leq n(x, s-1) + 2 \left(\left\lceil \left(\frac{x}{a_s} \right)^{1/b_s} \right\rceil - 1 \right) n(x, s-1) = \left(2 \left\lceil \left(\frac{x}{a_s} \right)^{1/b_s} \right\rceil - 1 \right) n(x, s-1).$$

This yields

$$n(x, s) \leq \prod_{j=1}^s \left(2 \left\lceil \left(\frac{x}{a_j} \right)^{1/b_j} \right\rceil - 1 \right).$$

If $j > j(x)$, i.e., $x \leq a_j$, then the factor

$$2 \left\lceil \left(\frac{x}{a_j} \right)^{1/b_j} \right\rceil - 1 = 2 \cdot 1 - 1 = 1.$$

Hence, we can restrict j in the last product to $\min(s, j(x))$ and obtain the desired upper bound on $n(x, s)$.

We turn to the lower bound on $n(x, s)$. Note that $x - a_s h^{b_s} > \alpha_s x$ for all $h \in \mathbb{N}$ with

$$h \leq \left\lceil \left(\frac{x(1 - \alpha_s)}{a_s} \right)^{1/b_s} \right\rceil - 1.$$

Hence

$$\begin{aligned} n(x, s) &\geq n(\alpha_s x, s-1) + 2n(\alpha_s x, s-1) \left(\left\lceil \left(\frac{x(1 - \alpha_s)}{a_s} \right)^{1/b_s} \right\rceil - 1 \right) \\ &= \left(2 \left\lceil \left(\frac{x(1 - \alpha_s)}{a_s} \right)^{1/b_s} \right\rceil - 1 \right) n(\alpha_s x, s-1). \end{aligned}$$

We now apply induction on s . For $s = 1$ we have

$$n(x, 1) = 2 \left\lceil (x/a_1)^{1/b_1} \right\rceil - 1 \geq 2 \left\lceil ((1 - \alpha_1)x/a_1)^{1/b_1} \right\rceil - 1,$$

as claimed. Then

$$\begin{aligned} n(x, s) &\geq \left(2 \left\lceil \left(\frac{x(1 - \alpha_s)}{a_s} \right)^{1/b_s} \right\rceil - 1 \right) \prod_{j=1}^{s-1} \left(2 \left\lceil \left(\frac{\alpha_s x}{a_j} (1 - \alpha_j) \prod_{k=j+1}^{s-1} \alpha_k \right)^{1/b_j} \right\rceil - 1 \right) \\ &\geq \prod_{j=1}^s \left(2 \left\lceil \left(\frac{x}{a_j} (1 - \alpha_j) \prod_{k=j+1}^s \alpha_k \right)^{1/b_j} \right\rceil - 1 \right) \\ &= \prod_{j=1}^{\min(s, j(x))} \left(2 \left\lceil \left(\frac{x}{a_j} (1 - \alpha_j) \prod_{k=j+1}^s \alpha_k \right)^{1/b_j} \right\rceil - 1 \right), \end{aligned}$$

as claimed. This completes the proof of the second point.

To prove the third point, it is enough to take $\alpha_j = (j-1)/j$. Then for $j = 1, 2, \dots, s$ we have

$$(1 - \alpha_j) \prod_{k=j+1}^s \alpha_k = \frac{\prod_{k=j+1}^s (k-1)}{j \prod_{k=j+1}^s k} = \frac{1}{s},$$

as claimed. This completes the proof of Lemma 1. \square

6 The proof of Theorem 1 for Λ^{all}

We are ready to prove Theorem 1 for the class Λ^{all} .

6.1 The proof of Point 1

From the second and third points of Lemma 1 with a fixed s we have

$$n(x, s) = \Theta(x^{B(s)}) \quad \text{as} \quad x \rightarrow \infty.$$

Therefore there are functions $c_1, c_2 : \mathbb{N} \rightarrow (0, \infty)$ such that

$$c_1(s) \log^{B(s)} \varepsilon^{-1} \leq n^{L_2\text{-app}, \Lambda^{\text{all}}}(\varepsilon, s) \leq c_2(s) \log^{B(s)} \varepsilon^{-1}$$

for ε tending to zero. This implies exponential convergence since

$$e^{L_2\text{-app}, \Lambda^{\text{all}}}(n, s) \leq q^{(n/c_2(s))^{1/B(s)}} \quad \text{with} \quad q = \exp(-1).$$

Hence, $p^*(s) \geq 1/B(s)$. On the other hand, if we have exponential convergence (6) then

$$n^{L_2\text{-app}, \Lambda^{\text{all}}}(\varepsilon, s) = \mathcal{O}\left(\log^{1/p(s)} \varepsilon^{-1}\right)$$

and $1/p(s) \geq B(s)$, or equivalently, $p(s) \leq 1/B(s)$. Hence, $p^*(s) = 1/B(s)$, as claimed in Point 1. The rest in this point is clear.

6.2 The proof of Point 2

Assume now that we have uniform exponential convergence. Then $e^{L_2\text{-app}, \Lambda^{\text{all}}}(n, s) \leq C(s) q^{(n/C_1(s))^p}$ implies for a fixed s that

$$n^{L_2\text{-app}, \Lambda^{\text{all}}}(\varepsilon, s) = \mathcal{O}(\log^{1/p} \varepsilon^{-1}) \quad \text{as} \quad \varepsilon \rightarrow 0.$$

Then $B(s) \leq 1/p$ for all s . Therefore $B \leq 1/p < \infty$ and $p^* \leq 1/B$. On the other hand, if $B < \infty$ then we can set $p(s) = 1/B$ and obtain uniform exponential convergence. Hence, $p^* \geq 1/B$, and therefore $p^* = 1/B$, as claimed. The rest of Point 2 is clear.

6.3 The proof of Point 3

PT means that

$$n^{L_2\text{-app}, \Lambda^{\text{all}}}(\varepsilon, s) \leq c s^{\tau_1} (1 + \log \varepsilon^{-1})^{\tau_2}.$$

This implies that

$$e^{L_2\text{-app}, \Lambda^{\text{all}}}(n) \leq e^{1-(n/c s^{\tau_1})^{1/\tau_2}}.$$

Hence, UEXP holds with $p = 1/\tau_2$. This also yields the equivalence between various notions of tractability with or without uniform exponential convergence.

6.4 The proof of Point 4

We first prove that WT implies $\lim_j a_j = \infty$. We use the first part of Lemma 1. For $\delta > 0$, take $x = (1 + \delta)(a_1 + \dots + a_s)$, or equivalently

$$\log \varepsilon^{-1} = \frac{x}{2} \log \omega^{-1} = \frac{\log \omega^{-1}}{2} (1 + \delta)(a_1 + a_2 + \dots + a_s).$$

Then

$$z_s := \frac{\log n(x, s)}{s + \log \varepsilon^{-1}} \geq \frac{s \log 3}{s + \frac{1}{2} x \log \omega^{-1}} = \frac{\log 3}{1 + \frac{1}{2} (1 + \delta) y_s \log \omega^{-1}},$$

where

$$y_s = \frac{a_1 + a_2 + \dots + a_s}{s}.$$

WT implies that $\lim_s z_s = 0$. This can hold only if $\lim_s y_s = \infty$ which implies that $\lim_j a_j = \infty$, as claimed.

Next, we need to prove that $\lim_j a_j = \infty$ implies WT. The eigenvalues of W_s are $\omega_{\mathbf{h}}$ for all $\mathbf{h} \in \mathbb{Z}^s$. Let the ordered eigenvalues of W_s be $\lambda_{s,n}$ for $n \in \mathbb{N}$ with $\lambda_{s,1} \geq \lambda_{s,2} \geq \lambda_{s,3} \geq \dots$. Obviously $\{\lambda_{s,n}\}_{n \in \mathbb{N}} = \{\omega_{\mathbf{h}}\}_{\mathbf{h} \in \mathbb{Z}^s}$. Therefore for any $\eta \in (0, 1)$ we have

$$n \lambda_{s,n}^\eta \leq \sum_{j=1}^{\infty} \lambda_{s,j}^\eta = \sum_{\mathbf{h} \in \mathbb{Z}^s} \omega_{\mathbf{h}}^\eta = \prod_{j=1}^s \left(1 + 2 \sum_{h=1}^{\infty} \omega^{\eta a_j h^{b_j}} \right).$$

Note that

$$\sum_{h=1}^{\infty} \omega^{\eta a_j h^{b_j}} \leq \sum_{h=1}^{\infty} \omega^{\eta a_j h} = \frac{\omega^{\eta a_j}}{1 - \omega^{\eta a_j}}.$$

This proves that

$$\lambda_{s,n} \leq \frac{\prod_{j=1}^s (1 + 2 \omega^{\eta a_j} / (1 - \omega^{\eta a_j}))^{1/\eta}}{n^{1/\eta}}. \quad (11)$$

Since $n^{L_2\text{-app}, \Lambda^{\text{all}}}(\varepsilon, s) = \min\{n : \lambda_{s,n+1} < \varepsilon^2\}$ we conclude that

$$n^{L_2\text{-app}, \Lambda^{\text{all}}}(\varepsilon, s) \leq \frac{\prod_{j=1}^s (1 + 2 \omega^{\eta a_j} / (1 - \omega^{\eta a_j}))}{\varepsilon^{2\eta}}.$$

Using $\log(1 + x) \leq x$ for $x \geq 0$, this yields

$$\log n^{L_2\text{-app}, \Lambda^{\text{all}}}(\varepsilon, s) \leq 2\eta \log \varepsilon^{-1} + 2 \sum_{j=1}^s c_j,$$

where

$$c_j = \frac{\omega^{\eta a_j}}{1 - \omega^{\eta a_j}}.$$

Note that $\lim_j a_j = \infty$ implies that $\lim_j c_j = 0$, and $\lim_s \sum_{j=1}^s c_j / s = 0$. Therefore

$$\limsup_{s + \log \varepsilon^{-1} \rightarrow \infty} \frac{\log n^{L_2\text{-app}, \Lambda^{\text{all}}}(\varepsilon, s)}{s + \log \varepsilon^{-1}} \leq 2\eta.$$

Since η can be arbitrarily small this proves that

$$\lim_{s + \log \varepsilon^{-1} \rightarrow \infty} \frac{\log n^{L_2\text{-app}, \Lambda^{\text{all}}}(\varepsilon, s)}{s + \log \varepsilon^{-1}} = 0.$$

Hence, WT holds for the class Λ^{all} , as claimed. The rest in this point follows from the previous results. This completes the proof of Point 4.

6.5 The proof of Points 5 and 6

For Point 5, it is enough to prove that PT implies SPT+UEXP. This will be done by showing that PT implies that $B < \infty$ and $\alpha^* > 0$. Then we show that $B < \infty$ and $\alpha^* > 0$ imply SPT+UEXP and obtain bounds on the exponent of SPT.

We know that PT implies UEXP and that UEXP implies that $B < \infty$. From the lower bound of Lemma 1 with $x = (1 + \delta)(a_1 + \dots + a_s)$ and from PT we have

$$3^s \leq n(x, s) \leq C s^{\tau_1} \left(1 + \frac{1 + \delta}{2} (\log \omega^{-1})(a_1 + \dots + a_s) \right)^{\tau_2} \quad \text{for all } s \in \mathbb{N}.$$

Since $a_1 \leq a_2 \leq a_3 \leq \dots$, this yields

$$s a_s \geq a_1 + \dots + a_s \geq \frac{2}{(1 + \delta) \log \omega^{-1}} \left[\left(\frac{3^s}{C s^{\tau_1}} \right)^{1/\tau_2} - 1 \right] \quad \text{for all } s \in \mathbb{N}.$$

Hence,

$$\alpha^* = \liminf_{s \rightarrow \infty} \frac{\log a_s}{s} \geq \frac{\log 3}{\tau_2} > 0,$$

as needed. This also shows that $\tau_2 \geq (\log 3)/\alpha^*$. Since this holds for all τ_2 for which we have SPT, we conclude that the exponent τ^* of SPT also satisfies $\tau^* \geq (\log 3)/\alpha^*$. Clearly, τ^* cannot be smaller than the reciprocal of the exponent p^* of UEXP. Hence, $\tau^* \geq B$. This completes this part of the proof as well as the proof of lower bounds on the exponent of SPT.

Assume now that $B < \infty$ and $\alpha^* \in (0, \infty]$. From (1) with $\delta \in (0, \alpha^*)$ we have

$$a_j \geq \exp(\delta j) \quad \text{for all } j \geq j_\delta^*.$$

Then

$$j(x) \leq \max \left(j_\delta^*, \frac{\log x}{\delta} \right).$$

For $x > a_1$, the upper bound on $n(x, s)$ from Lemma 1 yields

$$\begin{aligned} n(x, s) &\leq \prod_{j=1}^{\min(s, j(x))} \left(1 + 2 \left(\frac{x}{a_j} \right)^{1/b_j} \right) \\ &\leq \left[\prod_{j=1}^{\min(s, j(x))} \left(\frac{x}{a_j} \right)^{1/b_j} \right] 3^{\min(s, j(x))} \\ &\leq x^B \max(3^{j_\delta^*}, x^{(\log 3)/\delta}) \\ &\leq 3^{j_\delta^*} x^{B + (\log 3)/\delta}. \end{aligned} \tag{12}$$

Hence, SPT+UEXP holds, as claimed. Furthermore, since δ can be arbitrarily close to α^* , we conclude that the exponent of SPT satisfies

$$\tau^* \leq B + \frac{\log 3}{\alpha^*},$$

where for $\alpha^* = \infty$ we have $\frac{\log 3}{\alpha^*} = 0$. This completes the proof of Point 5 and of Point 6. The proof of the whole theorem for the class Λ^{all} is now completed. \square

7 Preliminaries for the class Λ^{std}

We state some preliminary observations which will be needed to prove Theorem 1 for the class Λ^{std} . Based on the definition of the set $\mathcal{A}(s, M)$ in (9) for $M > 1$, we will study approximating $f \in H(K_{s,a,b})$ by algorithms of the form

$$A_{n,s,M}(f)(\mathbf{x}) = \sum_{\mathbf{h} \in \mathcal{A}(s,M)} \left(\frac{1}{n} \sum_{k=1}^n f(\mathbf{x}_k) \exp(-2\pi i \mathbf{h} \cdot \mathbf{x}_k) \right) \exp(2\pi i \mathbf{h} \cdot \mathbf{x}), \quad (13)$$

where $\mathbf{x} \in [0, 1]^s$. Note that $A_{n,s,M}$ is a linear algorithm as in (5) with

$$\alpha_k(\mathbf{x}) = \frac{1}{n} \sum_{\mathbf{h} \in \mathcal{A}(s,M)} \exp(2\pi i \mathbf{h} \cdot (\mathbf{x} - \mathbf{x}_k))$$

and with $L_k(f) = f(\mathbf{x}_k)$ for deterministically chosen sample points $\mathbf{x}_k \in [0, 1]^s$ for $1 \leq k \leq n$. Hence, $L_k \in \Lambda^{\text{std}}$. The choice of M and \mathbf{x}_k will be given later.

We first study upper bounds on the worst-case error of $A_{n,s,M}$. The following analysis is similar to that in [5]. We have

$$\begin{aligned} (f - A_{n,s,M}(f))(\mathbf{x}) &= \sum_{\mathbf{h} \notin \mathcal{A}(s,M)} \widehat{f}(\mathbf{h}) \exp(2\pi i \mathbf{h} \cdot \mathbf{x}) \\ &\quad + \sum_{\mathbf{h} \in \mathcal{A}(s,M)} \left(\widehat{f}(\mathbf{h}) - \frac{1}{n} \sum_{k=1}^n f(\mathbf{x}_k) \exp(-2\pi i \mathbf{h} \cdot \mathbf{x}_k) \right) \exp(2\pi i \mathbf{h} \cdot \mathbf{x}). \end{aligned}$$

Using Parseval's identity we obtain

$$\begin{aligned} \|f - A_{n,s,M}(f)\|_{L_2([0,1]^s)}^2 &= \sum_{\mathbf{h} \notin \mathcal{A}(s,M)} |\widehat{f}(\mathbf{h})|^2 + \sum_{\mathbf{h} \in \mathcal{A}(s,M)} \left| \widehat{f}(\mathbf{h}) - \frac{1}{n} \sum_{k=1}^n f(\mathbf{x}_k) \exp(-2\pi i \mathbf{h} \cdot \mathbf{x}_k) \right|^2 \\ &= \sum_{\mathbf{h} \notin \mathcal{A}(s,M)} |\widehat{f}(\mathbf{h})|^2 \\ &\quad + \sum_{\mathbf{h} \in \mathcal{A}(s,M)} \left| \int_{[0,1]^s} f(\mathbf{x}) \exp(-2\pi i \mathbf{h} \cdot \mathbf{x}) d\mathbf{x} - \frac{1}{n} \sum_{k=1}^n f(\mathbf{x}_k) \exp(-2\pi i \mathbf{h} \cdot \mathbf{x}_k) \right|^2. \end{aligned} \quad (14)$$

We have

$$\sum_{\mathbf{h} \notin \mathcal{A}(s,M)} |\widehat{f}(\mathbf{h})|^2 = \sum_{\mathbf{h} \notin \mathcal{A}(s,M)} |\widehat{f}(\mathbf{h})|^2 \omega_{\mathbf{h}} \omega_{\mathbf{h}}^{-1} \leq \frac{1}{M} \|f\|_{H(K_{s,a,b})}^2. \quad (15)$$

For the second term in (14), we make a specific choice for the points $\mathbf{x}_1, \dots, \mathbf{x}_n$ used in the algorithm $A_{n,s,M}$. Namely, we take \mathbf{x}_j 's from a regular grid with different mesh-sizes for successive variables. Such regular grids have already been studied in [2, 4]. We now recall their definition. For $s \in \mathbb{N}$, a regular grid with mesh-sizes $m_1, \dots, m_s \in \mathbb{N}$ is defined as the point set

$$\mathcal{G}_{n,s} = \{(k_1/m_1, \dots, k_s/m_s) : k_j = 0, 1, \dots, m_j - 1 \text{ for all } j = 1, 2, \dots, s\},$$

where $n = \prod_{j=1}^s m_j$ is the cardinality of $\mathcal{G}_{n,s}$. By $\mathcal{G}_{n,s}^\perp$ we denote the dual of $\mathcal{G}_{n,s}$, i.e.,

$$\mathcal{G}_{n,s}^\perp = \{\mathbf{h} \in \mathbb{Z}^s : h_j \equiv 0 \pmod{m_j} \text{ for all } j = 1, 2, \dots, s\}.$$

We will make use of the following result whose easy proof is omitted.

Lemma 2 *Let $\mathcal{G}_{n,s} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ be defined as above. For any $f \in H(K_{s,\mathbf{a},\mathbf{b}})$ we have*

$$\left| \int_{[0,1]^s} f(\mathbf{x}) \, d\mathbf{x} - \frac{1}{n} \sum_{k=1}^n f(\mathbf{x}_k) \right| = \left| \sum_{\mathbf{h} \in \mathcal{G}_{n,s}^\perp \setminus \{\mathbf{0}\}} \widehat{f}(\mathbf{h}) \right|.$$

For $\mathbf{h} \in \mathbb{Z}^s$ define $f_{\mathbf{h}}(\mathbf{x}) := f(\mathbf{x}) \exp(-2\pi i \mathbf{h} \cdot \mathbf{x})$. Note that with f also $f_{\mathbf{h}}$ belongs to $H(K_{s,\mathbf{a},\mathbf{b}})$ and that $\widehat{f_{\mathbf{h}}}(\mathbf{k}) = \widehat{f}(\mathbf{h} + \mathbf{k})$. From Lemma 2 we obtain

$$\begin{aligned} \left| \int_{[0,1]^s} f_{\mathbf{h}}(\mathbf{x}) \, d\mathbf{x} - \frac{1}{n} \sum_{k=1}^n f_{\mathbf{h}}(\mathbf{x}_k) \right|^2 &= \left| \sum_{\mathbf{l} \in \mathcal{G}_{n,s}^\perp \setminus \{\mathbf{0}\}} \widehat{f_{\mathbf{h}}}(\mathbf{l}) \right|^2 = \left| \sum_{\mathbf{l} \in \mathcal{G}_{n,s}^\perp \setminus \{\mathbf{0}\}} \widehat{f}(\mathbf{l} + \mathbf{h}) \right|^2 \\ &\leq \left(\sum_{\mathbf{l} \in \mathcal{G}_{n,s}^\perp \setminus \{\mathbf{0}\}} \left| \widehat{f}(\mathbf{l} + \mathbf{h}) \right|^2 \omega_{\mathbf{h}+\mathbf{l}}^{-1} \right) \left(\sum_{\mathbf{l} \in \mathcal{G}_{n,s}^\perp \setminus \{\mathbf{0}\}} \omega_{\mathbf{h}+\mathbf{l}} \right) \\ &\leq \|f\|_{H(K_{s,\mathbf{a},\mathbf{b}})}^2 \left(\sum_{\mathbf{l} \in \mathcal{G}_{n,s}^\perp \setminus \{\mathbf{0}\}} \omega_{\mathbf{h}+\mathbf{l}} \right). \end{aligned}$$

Therefore, and using (14) and (15) for any $f \in H(K_{s,\mathbf{a},\mathbf{b}})$ with $\|f\|_{H(K_{s,\mathbf{a},\mathbf{b}})} \leq 1$, we obtain

$$\|f - A_{n,s,M}(f)\|_{L_2([0,1]^s)}^2 \leq \frac{1}{M} + \sum_{\mathbf{h} \in \mathcal{A}(s,M)} \sum_{\mathbf{l} \in \mathcal{G}_{n,s}^\perp \setminus \{\mathbf{0}\}} \omega_{\mathbf{h}+\mathbf{l}}. \quad (16)$$

It is easy to see that

$$|\ell|^b \leq 2^b (|h + \ell|^b + |h|^b)$$

for any $h, \ell \in \mathbb{Z}$ and any $b \in \mathbb{N}$. For $\mathbf{h} \in \mathcal{A}(s, M)$ this implies

$$\omega_{\mathbf{h}+\mathbf{l}} = \omega^{\sum_{j=1}^s a_j |h_j + \ell_j|^{b_j}} \leq \omega^{\sum_{j=1}^s 2^{-b_j} a_j |\ell_j|^{b_j}} \omega^{-\sum_{j=1}^s a_j |h_j|^{b_j}} \leq \omega^{\sum_{j=1}^s 2^{-b_j} a_j |\ell_j|^{b_j}} M. \quad (17)$$

Using (16), (17) and Lemma 1 with $x = (\log M)/(\log \omega^{-1})$, we obtain for any $f \in H(K_{s,\mathbf{a},\mathbf{b}})$ with $\|f\|_{H(K_{s,\mathbf{a},\mathbf{b}})} \leq 1$,

$$\begin{aligned} \|f - A_{n,s,M}(f)\|_{L_2([0,1]^s)}^2 &\leq \frac{1}{M} + M |\mathcal{A}(s, M)| \sum_{\mathbf{l} \in \mathcal{G}_{n,s}^\perp \setminus \{\mathbf{0}\}} \omega^{\sum_{j=1}^s 2^{-b_j} a_j |\ell_j|^{b_j}} \\ &\leq \frac{1}{M} + M \left(\prod_{j=1}^s \left(1 + 2 \left(\frac{\log M}{a_j \log \omega^{-1}} \right)^{1/b_j} \right) \right) F_n, \end{aligned}$$

where

$$F_n := \sum_{\mathbf{l} \in \mathcal{G}_{n,s}^\perp \setminus \{\mathbf{0}\}} \omega^{\sum_{j=1}^s 2^{-b_j} a_j |\ell_j|^{b_j}}.$$

This means that

$$[e^{L_2\text{-app}}(H(K_{s,\mathbf{a},\mathbf{b}}), A_{n,s,M})]^2 \leq \frac{1}{M} + M \left(\prod_{j=1}^s \left(1 + 2 \left(\frac{\log M}{a_j \log \omega^{-1}} \right)^{1/b_j} \right) \right) F_n. \quad (18)$$

Furthermore,

$$\begin{aligned} \prod_{j=1}^s \left(1 + 2 \left(\frac{\log M}{a_j \log \omega^{-1}} \right)^{1/b_j} \right) &\leq 2^s \prod_{j=1}^s \left(1 + \left(\frac{\log M}{\log \omega^{-1}} \right)^{1/b_j} \right) \\ &\leq 2^s \prod_{j=1}^s \left(1 + \log^{-1/b_j} \omega^{-1} \right) \prod_{j=1}^s \left(1 + \log^{1/b_j} M \right). \end{aligned}$$

Since M is assumed to be at least 1, we can bound $1 + \log^{1/b_j} M \leq 2M^{1/b_j}$, and obtain

$$\prod_{j=1}^s \left(1 + 2 \left(\frac{\log M}{a_j \log \omega^{-1}} \right)^{1/b_j} \right) \leq 4^s M^{B(s)} \prod_{j=1}^s \left(1 + \log^{-1/b_j} \omega^{-1} \right),$$

where, as in the previous sections, $B(s) := \sum_{j=1}^s b_j^{-1}$. Plugging this into (18), we obtain

$$[e^{L_2\text{-app}}(H(K_{s,\mathbf{a},\mathbf{b}}), A_{n,s,M})]^2 \leq \frac{1}{M} + M^{B(s)+1} D(s, \omega, \mathbf{b}) F_n, \quad (19)$$

where

$$D(s, \omega, \mathbf{b}) := 4^s \prod_{j=1}^s \left(1 + \log^{-1/b_j} \omega^{-1} \right).$$

8 The proof of Theorem 1 for Λ^{std}

We now present the proofs for the successive points of Theorem 1 for the class Λ^{std} .

8.1 The proof of Point 1

The following proposition will be helpful.

Proposition 1 *For $s \in \mathbb{N}$ and $\varepsilon \in (0, 1)$ define*

$$m = \max_{j=1,2,\dots,s} \left[\left(\frac{4^{b_j} \log \left(1 + \frac{2s}{\log(1+\eta^2)} \right)}{a_j \log \omega^{-1}} \right)^{B(s)} \right],$$

where

$$\eta = \left(\frac{\varepsilon^2}{2D(s, \omega, \mathbf{b})^{\frac{1}{B(s)+2}}} \right)^{\frac{B(s)+2}{2}}.$$

Let $\mathcal{G}_{n,s}^*$ be a regular grid with mesh-sizes m_1, m_2, \dots, m_s given by

$$m_j := \lfloor m^{1/(B(s) \cdot b_j)} \rfloor \quad \text{for } j = 1, 2, \dots, s \quad \text{and} \quad n = \prod_{j=1}^s m_j.$$

Then for $M = 2/\varepsilon^2$ we have

$$e^{L_2\text{-app}}(H(K_{s,\mathbf{a},\mathbf{b}}), A_{n,s,M}) \leq \varepsilon, \quad \text{and} \quad n = \mathcal{O}\left(\log^{B(s)}(1 + \varepsilon^{-1})\right)$$

with the factor in the \mathcal{O} notation independent of ε^{-1} but dependent on s .

Proof. We can write

$$F_n = \sum_{\mathbf{l} \in \mathcal{G}_{n,s}^\perp \setminus \{\mathbf{0}\}} \omega^{\sum_{j=1}^s 2^{-b_j} a_j |\ell_j|^{b_j}} = -1 + \prod_{j=1}^s \left(1 + 2 \sum_{h=1}^{\infty} \omega^{a_j 2^{-b_j} (m_j h)^{b_j}}\right).$$

Since $\lfloor x \rfloor \geq x/2$ for all $x \geq 1$, we have

$$|m_j h_j|^{b_j} \geq (|h_j|/2)^{b_j} m^{1/B(s)} \quad \text{for all} \quad j = 1, 2, \dots, s.$$

Hence,

$$F_n \leq -1 + \prod_{j=1}^s \left(1 + 2 \sum_{h=1}^{\infty} \omega^{m^{1/B(s)} a_j 4^{-b_j} h^{b_j}}\right).$$

Since $b_j \geq 1$ we further estimate

$$\sum_{h=1}^{\infty} \omega^{m^{1/B(s)} a_j 4^{-b_j} h^{b_j}} \leq \sum_{h=1}^{\infty} \omega^{m^{1/B(s)} a_j 4^{-b_j} h} = \frac{\omega^{m^{1/B(s)} a_j 4^{-b_j}}}{1 - \omega^{m^{1/B(s)} a_j 4^{-b_j}}}.$$

From the definition of m we have

$$\frac{\omega^{m^{1/B(s)} a_j 4^{-b_j}}}{1 - \omega^{m^{1/B(s)} a_j 4^{-b_j}}} \leq \frac{\log(1 + \eta^2)}{2s} \quad \text{for all} \quad j = 1, 2, \dots, s.$$

This proves

$$F_n \leq -1 + \left(1 + \frac{\log(1 + \eta^2)}{s}\right)^s \leq -1 + \exp(\log(1 + \eta^2)) = \eta^2. \quad (20)$$

Now, plugging this into (19), we obtain

$$[e^{L_2\text{-app}}(H(K_{s,\mathbf{a},\mathbf{b}}), A_{n,s,M})]^2 \leq \frac{1}{M} + M^{B(s)+1} D(s, \omega, \mathbf{b}) \eta^2. \quad (21)$$

Note that

$$\frac{1}{D(s, \omega, \mathbf{b})^{\frac{1}{B(s)+2}} \eta^{\frac{2}{B(s)+2}}} = \frac{2}{\varepsilon^2} \geq 1.$$

Hence we are allowed to choose

$$M = \frac{1}{D(s, \omega, \mathbf{b})^{\frac{1}{B(s)+2}} \eta^{\frac{2}{B(s)+2}}},$$

which yields, inserting into (21),

$$[e^{L_2\text{-app}}(H(K_{s,\mathbf{a},\mathbf{b}}), A_{n,s,M})]^2 \leq 2D(s, \omega, \mathbf{b})^{\frac{1}{B(s)+2}} \eta^{\frac{2}{B(s)+2}} = \varepsilon^2,$$

as claimed.

It remains to verify that n is of the order stated in the proposition. Note that

$$n = \prod_{j=1}^s m_j = \prod_{j=1}^s \lfloor m^{1/(B(s) \cdot b_j)} \rfloor \leq m^{\frac{1}{B(s)} \sum_{j=1}^s 1/b_j} = m.$$

However, as pointed out in [4],

$$m = \mathcal{O} \left(\log^{B(s)} (1 + \eta^{-1}) \right),$$

as η tends to zero. From this, it is easy to see that we indeed have

$$m = \mathcal{O} \left(\log^{B(s)} (1 + \varepsilon^{-1}) \right),$$

which concludes the proof of Proposition 1. \square

To show Point 1 for the class Λ^{std} , we conclude from Proposition 1 that

$$n^{L_2\text{-app}, \Lambda^{\text{std}}}(\varepsilon, s) = \mathcal{O} \left(\log^{B(s)} (1 + \varepsilon^{-1}) \right).$$

This implies that we indeed have exponential convergence for Λ^{std} for all \mathbf{a} and \mathbf{b} , with $p(s) = 1/B(s)$, and thus $p^*(s) \geq 1/B(s)$. On the other hand, note that obviously $e^{L_2\text{-app}, \Lambda^{\text{std}}}(n, s) \geq e^{L_2\text{-app}, \Lambda^{\text{all}}}(n, s)$, hence the rate of exponential convergence for Λ^{std} cannot be larger than for Λ^{all} which is $1/B(s)$. Thus, also for the class Λ^{std} we have $p^*(s) = 1/B(s)$. The rest of Point 1 is clear.

8.2 The proof of Point 2

We turn to Point 2 for the class Λ^{std} . Suppose first that \mathbf{a} is an arbitrary sequence and that \mathbf{b} is such that

$$B = \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty.$$

Then we can replace $B(s)$ by B in Proposition 1, and we obtain

$$n^{L_2\text{-app}, \Lambda^{\text{std}}}(\varepsilon, s) = \mathcal{O} \left(\log^B (1 + \varepsilon^{-1}) \right),$$

hence uniform exponential convergence with $p^* \geq 1/B$ holds. On the other hand, if we have uniform exponential convergence for Λ^{std} , this implies uniform exponential convergence for Λ^{all} , which in turn implies that $B < \infty$ and that $p^* \leq 1/B$. The rest of Point 2 follows immediately.

8.3 The proof of Point 3

The proof of Point 3 follows by the same arguments as for Λ^{all} .

8.4 The proof of Point 4

We now prove the first part of Point 4 for the class Λ^{std} . Assume that WT holds for the class Λ^{std} . Then WT also holds for the class Λ^{all} and this implies that $\lim_j a_j = \infty$, as claimed.

Assume now that $\lim_j a_j = \infty$. We use [9, Theorem 26.18]. In particular, this theorem states that if the ordered eigenvalues $\lambda_{s,n}$'s of W_s satisfy

$$\lambda_{s,n} \leq \frac{M_{s,\tau}^2}{n^{2\tau}} \quad \text{for all } n \in \mathbb{N}, \quad (22)$$

for some positive $M_{s,\tau}$ and $\tau > \frac{1}{2}$ then there is a semi-constructive algorithm² such that

$$e^{L_2\text{-app}, \Lambda^{\text{std}}}(n+2, s) \leq \frac{M_{s,\tau} C(\tau)}{n^{\tau(2\tau/(2\tau+1))}} \quad \text{for all } n \in \mathbb{N} \quad (23)$$

where $C(\tau)$ is given explicitly in [9, Theorem 26.18]. However, the form of $C(\tau)$ is not important for our consideration.

For $\eta \in (0, 1)$, let $\tau = 1/(2\eta) > \frac{1}{2}$. We stress that τ can be arbitrarily large if we take sufficiently small η . We already showed in the proof for the class Λ^{all} , see (11), that we can take

$$M_{s,\tau} = \prod_{j=1}^s (1 + 2c_j)^\tau < \infty \quad \text{with} \quad c_j = \frac{\omega^{a_j/(2\tau)}}{1 - \omega^{a_j/(2\tau)}}.$$

Furthermore, we know that $\lim_j a_j = \infty$ implies that $\lim_s \sum_{j=1}^s c_j/s = 0$.

From (23) we obtain

$$n^{L_2\text{-app}, \Lambda^{\text{std}}}(\varepsilon, s) \leq 3 + (M_{s,\tau} C(\tau))^{(1+1/(2\tau))/\tau} \varepsilon^{-(1+1/(2\tau))/\tau}.$$

This yields that

$$\limsup_{s+\log \varepsilon^{-1} \rightarrow \infty} \frac{\log n^{L_2\text{-app}, \Lambda^{\text{std}}}(\varepsilon, s)}{s + \log \varepsilon^{-1}} \leq \left(1 + \frac{1}{2\tau}\right) \frac{1}{\tau} \left(1 + \limsup_{s \rightarrow \infty} \frac{\log M_{s,\tau}}{s}\right).$$

Since $(\log M_{s,\tau})/s \leq 2\tau \sum_{j=1}^s c_j/s$ tends to zero as $s \rightarrow \infty$, we have

$$\limsup_{s+\log \varepsilon^{-1} \rightarrow \infty} \frac{\log n^{L_2\text{-app}, \Lambda^{\text{std}}}(\varepsilon, s)}{s + \log \varepsilon^{-1}} \leq \left(1 + \frac{1}{2\tau}\right) \frac{1}{\tau}.$$

Since τ can be arbitrarily large this proves that

$$\lim_{s+\log \varepsilon^{-1} \rightarrow \infty} \frac{\log n^{L_2\text{-app}, \Lambda^{\text{std}}}(\varepsilon, s)}{s + \log \varepsilon^{-1}} = 0.$$

This means that WT holds for the class Λ^{std} , as claimed.

We turn to the second part of Point 4 for the class Λ^{std} . This point easily follows from the already proved facts that WT holds iff $\lim_j a_j = \infty$ and UEXP holds iff $B < \infty$.

²By semi-constructive we mean that this algorithm can be constructed after a few random selections of sample points, more can be found in [9].

8.5 The proof of Point 5

Suppose that PT holds for the class Λ^{std} . Then PT holds for the class Λ^{all} . By Point 5 for the class Λ^{all} , which has already been proved, this implies SPT+UEXP for the class Λ^{all} which in turn implies that $B < \infty$ and $\alpha^* > 0$ by Point 6 for the class Λ^{all} . This implies SPT+UEXP for the class Λ^{std} as will be shown in the subsequent Section 8.6. The rest of this point is clear.

8.6 The proof of Point 6

The necessity of the conditions for SPT+UEXP on \mathbf{b} and \mathbf{a} stated in Point 6 for the class Λ^{std} follows from the same conditions for the class Λ^{all} and the fact that the information complexity for Λ^{std} cannot be smaller than for Λ^{all} .

To prove the sufficiency of the conditions for SPT+UEXP on \mathbf{b} and \mathbf{a} stated in Point 6 we analyze the algorithm $A_{n,s,M}$ given by (13), where the sample points \mathbf{x}_k are from the regular grid $\mathcal{G}_{n,s}$ with mesh-sizes

$$m_j = 2 \left\lceil \left(\frac{\log M}{a_j^\beta \log \omega^{-1}} \right)^{1/b_j} \right\rceil - 1 \quad \text{for all } j = 1, 2, \dots, s.$$

Here $M > 1$ and $\beta \in (0, 1)$. Note that $m_j \geq 1$ and is always an odd number. Furthermore $m_j = 1$ if $a_j \geq ((\log M)/(\log \omega^{-1}))^{1/\beta}$. Assume that $\alpha^* \in (0, \infty]$. Since for all $\delta \in (0, \alpha^*)$ we have

$$a_j \geq \exp(\delta j) \quad \text{for all } j \geq j_\delta^*,$$

see (1), we conclude that

$$j \geq j_{\beta,\delta}^* := \max \left(j_\delta^*, \frac{\log(((\log M)/(\log \omega^{-1}))^{1/\beta})}{\delta} \right) \quad \text{implies } m_j = 1.$$

From (16) we have

$$e_{n,s}^2 := [e^{L_2\text{-app}}(H(K_{s,\mathbf{a},\mathbf{b}}), A_{n,s,M})]^2 \leq \frac{1}{M} + \sum_{\mathbf{h} \in \mathcal{A}(s,M)} \sum_{\mathbf{l} \in \mathcal{G}_{n,s}^+ \setminus \{\mathbf{0}\}} \omega_{\mathbf{h}+\mathbf{l}}.$$

We now estimate

$$\sum_{\mathbf{l} \in \mathcal{G}_{n,s}^+ \setminus \{\mathbf{0}\}} \omega_{\mathbf{h}+\mathbf{l}} = \sum_{\emptyset \neq \mathbf{u} \subseteq \{1, \dots, s\}} \prod_{j \in \mathbf{u}} \left(\sum_{\ell_j \in \mathbb{Z} \setminus \{0\}} \omega^{a_j |h_j + m_j \ell_j|^{b_j}} \right) \prod_{j \notin \mathbf{u}} \omega^{a_j |h_j|^{b_j}},$$

where we separated the cases for $\ell_j \in \mathbb{Z} \setminus \{0\}$ and $\ell_j = 0$. We estimate the second product by one so that

$$\sum_{\mathbf{l} \in \mathcal{G}_{n,s}^+ \setminus \{\mathbf{0}\}} \omega_{\mathbf{h}+\mathbf{l}} \leq \sum_{\emptyset \neq \mathbf{u} \subseteq \{1, \dots, s\}} \prod_{j \in \mathbf{u}} \left(\sum_{\ell \in \mathbb{Z} \setminus \{0\}} \omega^{a_j |h_j + m_j \ell|^{b_j}} \right).$$

We now show that for $\mathbf{h} \in \mathcal{A}(s, M)$ we have $|h_j| < (m_j + 1)/2$ for all $j = 1, 2, \dots, s$. Indeed, the vector \mathbf{h} satisfies $\prod_{j=1}^s \omega^{-a_j |h_j|^{b_j}} < M$, and since each factor is at least one we have $\omega^{-a_j |h_j|^{b_j}} < M$ for all j , which is equivalent to

$$|h_j| < \left(\frac{\log M}{a_j \log \omega^{-1}} \right)^{1/b_j} \leq \left(\frac{\log M}{a_j^\beta \log \omega^{-1}} \right)^{1/b_j} \leq \frac{m_j + 1}{2},$$

as claimed.

In particular, if $m_j = 1$ then $h_j = 0$ and

$$\sum_{\ell \in \mathbb{Z} \setminus \{0\}} \omega^{a_j |h_j + m_j \ell|^{b_j}} = 2 \sum_{\ell=1}^{\infty} \omega^{a_j \ell^{b_j}} \leq 2 \sum_{\ell=1}^{\infty} \omega^{a_j \ell} = \frac{2 \omega^{a_j}}{1 - \omega^{a_j}} \leq \frac{2 \omega^{a_j}}{1 - \omega}. \quad (24)$$

Let $m_j \geq 3$. Then $|h_j| < (m_j + 1)/2$. Since both $|h_j|$ and $(m_j + 1)/2$ are positive integers, we conclude that $|h_j| \leq (m_j + 1)/2 - 1 = (m_j - 1)/2$ and $\ell \neq 0$ implies

$$|h_j + m_j \ell| \geq m_j |\ell| - |h_j| \geq \frac{m_j + 1}{2} |\ell|.$$

Therefore

$$\sum_{\ell \in \mathbb{Z} \setminus \{0\}} \omega^{a_j |h_j + m_j \ell|^{b_j}} \leq 2 \sum_{\ell=1}^{\infty} \omega^{a_j [(m_j + 1)/2]^{b_j} \ell^{b_j}} \leq \frac{2 \omega^{a_j [(m_j + 1)/2]^{b_j}}}{1 - \omega}. \quad (25)$$

The inequalities (24) and (25) can be combined as

$$\beta_j := \sum_{\ell \in \mathbb{Z} \setminus \{0\}} \omega^{a_j |h_j + m_j \ell|^{b_j}} \leq \frac{2 \omega^{a_j [(m_j + 1)/2]^{b_j}}}{1 - \omega}.$$

Note that

$$\sum_{\emptyset \neq \mathbf{u} \subseteq \{1, \dots, s\}} \prod_{j \in \mathbf{u}} \left(\sum_{\ell \in \mathbb{Z} \setminus \{0\}} \omega^{a_j |h_j + m_j \ell|^{b_j}} \right) = -1 + \sum_{\mathbf{u} \subseteq \{1, \dots, s\}} \prod_{j \in \mathbf{u}} \beta_j = -1 + \prod_{j=1}^s (1 + \beta_j).$$

Consequently,

$$e_{n,s}^2 \leq \frac{1}{M} + |\mathcal{A}(s, M)| \left(-1 + \prod_{j=1}^s \left(1 + \frac{2 \omega^{a_j [(m_j + 1)/2]^{b_j}}}{1 - \omega} \right) \right).$$

Using $\log(1 + x) \leq x$ we obtain

$$\log \left[\prod_{j=1}^s \left(1 + \frac{2 \omega^{a_j [(m_j + 1)/2]^{b_j}}}{1 - \omega} \right) \right] \leq \frac{2}{1 - \omega} \sum_{j=1}^s \omega^{a_j [(m_j + 1)/2]^{b_j}}.$$

From the definition of m_j we have $a_j [(m_j + 1)/2]^{b_j} \geq a_j^{1-\beta} (\log M) / \log \omega^{-1}$. Therefore

$$\omega^{a_j [(m_j + 1)/2]^{b_j}} \leq \omega^{a_j^{1-\beta} (\log M) / \log \omega^{-1}} = \left(\frac{1}{M} \right)^{a_j^{1-\beta}}.$$

Since $a_j \geq 1$ for $j \leq j_{\beta,\delta}^* - 1$ and $a_j \geq \exp(\delta j)$ for $j \geq j_{\beta,\delta}^*$ we obtain

$$\gamma := \frac{2}{1-\omega} \sum_{j=1}^s \omega^{a_j[(m_j+1)/2]^{b_j}} \leq \frac{2}{1-\omega} \left(\frac{j_{\beta,\delta}^* - 1}{M} + \sum_{j=j_{\beta,\delta}^*}^{\infty} \left(\frac{1}{M} \right)^{\exp((1-\beta)\delta j)} \right) = \frac{C_{\beta,\delta}}{M},$$

where

$$C_{\beta,\delta} := \frac{2}{1-\omega} \left(j_{\beta,\delta}^* - 1 + \sum_{j=j_{\beta,\delta}^*}^{\infty} \left(\frac{1}{M} \right)^{\exp((1-\beta)\delta j)-1} \right) < \infty.$$

Note that for $M \geq C_{\beta,\delta}$ we have $\gamma \leq 1$.

Using convexity we easily check that $-1 + \exp(\gamma) \leq (e-1)\gamma$ for all $\gamma \in [0, 1]$. Thus for $M \geq C_{\beta,\delta}$ we obtain

$$\begin{aligned} -1 + \prod_{j=1}^s \left(1 + \frac{2\omega^{a_j[(m_j+1)/2]^{b_j}}}{1-\omega} \right) &\leq -1 + \exp \left(\frac{2}{1-\omega} \sum_{j=1}^s \omega^{a_j[(m_j+1)/2]^{b_j}} \right) \\ &= -1 + \exp(\gamma) \leq (e-1)\gamma \\ &\leq \frac{C_{\beta,\delta}(e-1)}{M}. \end{aligned}$$

We now turn to $|\mathcal{A}(s, M)|$ which was already estimated in the proof for the class Λ^{all} , see (12). We have

$$|\mathcal{A}(s, M)| \leq 3^{j_{\beta,\delta}^*} \left(1 + \frac{\log M}{\log \omega^{-1}} \right)^{B+(\log 3)/\delta}.$$

Therefore

$$e_{n,s}^2 \leq \frac{1}{M} \left[1 + C_{\beta,\delta}(e-1)3^{j_{\beta,\delta}^*} \left(1 + \frac{\log M}{\log \omega^{-1}} \right)^{B+(\log 3)/\delta} \right] \leq \frac{D_{\beta,\delta}}{\sqrt{M}},$$

where

$$D_{\beta,\delta} := \sup_{x \geq C_{\beta,\delta}} \left(\frac{1}{\sqrt{x}} + \frac{C_{\beta,\delta}(e-1)3^{j_{\beta,\delta}^*}}{\sqrt{x}} \left(1 + \frac{\log x}{\log \omega^{-1}} \right)^{B+(\log 3)/\delta} \right) < \infty.$$

Hence for

$$M = \max(C_{\beta,\delta}, D_{\beta,\delta}^2 \varepsilon^{-4})$$

we have

$$e_{n,s} = e^{L_2\text{-app}}(H(K_{s,\mathbf{a},\mathbf{b}}), A_{n,s,M}) \leq \varepsilon.$$

We estimate the number n of function values used by the algorithm $A_{n,s,M}$. We have

$$\begin{aligned} n &= \prod_{j=1}^s m_j = \prod_{j=1}^{\min(s, j_{\beta,\delta}^*)} m_j \leq \prod_{j=1}^{\min(s, j_{\beta,\delta}^*)} \left(1 + 2 \left(\frac{\log M}{a_j^\beta \log \omega^{-1}} \right)^{1/b_j} \right) \\ &\leq 3^{j_{\beta,\delta}^*} \left(\frac{\log M}{\log \omega^{-1}} \right)^B \leq 3^{j_{\beta,\delta}^*} \left(\frac{\log M}{\log \omega^{-1}} \right)^{B+(\log 3)/(\beta \delta)} \end{aligned}$$

$$= \mathcal{O} \left((1 + \log \varepsilon^{-1})^{B + (\log 3)/(\beta \delta)} \right),$$

where the factor in the big \mathcal{O} notation depends only on β and δ . This proves SPT+UEXP with

$$\tau = B + \frac{\log 3}{\beta \delta}.$$

Since β can be arbitrarily close to one, and δ can be arbitrarily close to α^* , the exponent τ^* of SPT is at most

$$B + \frac{\log 3}{\alpha^*},$$

where for $\alpha^* = \infty$ we have $\frac{\log 3}{\alpha^*} = 0$. This completes the proof of Theorem 1 for the class Λ^{std} .

9 Relations to multivariate integration

Multivariate integration

$$\text{INT}_s(f) = \int_{[0,1]^s} f(\mathbf{x}) \, d\mathbf{x}$$

for f from the Korobov space $H(K_{s,\mathbf{a},\mathbf{b}})$ was studied in [4]. It is easy to see that multivariate approximation is not easier than multivariate integration, see e.g., [6]. More precisely, for any algorithm $A_{n,s}(f) = \sum_{k=1}^n \alpha_k f(\mathbf{x}_k)$ for multivariate approximation using the nodes $\mathbf{x}_1, \dots, \mathbf{x}_n \in [0,1]^s$ and $\alpha_k \in L_2([0,1]^s)$, define $\beta_k := \int_{[0,1]^s} \alpha_k(\mathbf{x}) \, d\mathbf{x}$ and the algorithm

$$A_{n,s}^{\text{int}}(f) = \sum_{k=1}^n \beta_k f(\mathbf{x}_k)$$

for multivariate integration. Then

$$\begin{aligned} \left| \int_{[0,1]^s} f(\mathbf{x}) \, d\mathbf{x} - A_{n,s}^{\text{int}}(f) \right| &= \left| \int_{[0,1]^s} \left(f(\mathbf{x}) - \sum_{k=1}^n \alpha_k(\mathbf{x}) f(\mathbf{x}_k) \right) \, d\mathbf{x} \right| \\ &\leq \left(\int_{[0,1]^s} \left(f(\mathbf{x}) - \sum_{k=1}^n \alpha_k(\mathbf{x}) f(\mathbf{x}_k) \right)^2 \, d\mathbf{x} \right)^{1/2} \\ &= \|f - A_{n,s}(f)\|_{L_2([0,1]^s)}. \end{aligned}$$

This proves that for the worst-case error for integration we have

$$e(H(K_{s,\mathbf{a},\mathbf{b}}), A_{n,s}^{\text{int}}) := \sup_{\substack{f \in H(K_{s,\mathbf{a},\mathbf{b}}) \\ \|f\|_{H(K_{s,\mathbf{a},\mathbf{b}})} \leq 1}} \left| \int_{[0,1]^s} f(\mathbf{x}) \, d\mathbf{x} - \sum_{k=1}^n \beta_k f(\mathbf{x}_k) \right| \leq e^{L_2\text{-app}}(H(K_{s,\mathbf{a},\mathbf{b}}), A_{n,s}).$$

Since this holds for all algorithms $A_{n,s}$ we conclude that

$$e^{\text{int}}(n, s) := \inf_{A_{n,s}^{\text{int}}} e(H(K_{s,\mathbf{a},\mathbf{b}}), A_{n,s}^{\text{int}}) \leq e^{\text{app}}(n, s) := e^{L_2\text{-app}, \Lambda^{\text{std}}}(n, s). \quad (26)$$

Here $e^{\text{int}}(n, s)$ and $e^{\text{app}}(n, s)$ are the n th minimal worst-case errors for multivariate integration and approximation in $H(K_{s, \mathbf{a}, \mathbf{b}})$, respectively. Furthermore for $n = 0$ we have equality,

$$e^{\text{int}}(0, s) = e^{\text{app}}(0, s) = 1.$$

From these observations it follows that for $\varepsilon \in (0, 1)$ and $s \in \mathbb{N}$ we have

$$n^{\text{int}}(\varepsilon, s) \leq n^{L_2\text{-app}, \Lambda^{\text{std}}}(\varepsilon, s), \quad (27)$$

where $n^{\text{int}}(\varepsilon, s)$ is the information complexity for the integration problem.

Obviously, for multivariate integration only the class Λ^{std} makes sense. The inequalities (26) and (27) mean that all positive results for multivariate approximation and the class Λ^{std} also hold for multivariate integration. In particular, the following facts hold:

- Exponential convergence holds for multivariate integration for arbitrary \mathbf{a} and \mathbf{b} with the largest rate $p^{\text{int}}(s) \geq 1/B(s)$. Although only uniform exponential convergence was considered in [4], the proof presented there allows to conclude that we have $p^{\text{int}}(s) = 1/B(s)$.
- Uniform convergence holds for multivariate integration iff $B < \infty$ and the largest rate $[p^{\text{int}}]^* = 1/B$, as for multivariate approximation. This was shown in [4].
- Polynomial tractability and strong polynomial tractability for multivariate integration were studied in [4], where it was shown that they are equivalent and hold iff $B < \infty$ and a_j 's are exponentially growing with j . These conditions are the same as for multivariate approximation.

The exponent $[\tau^{\text{int}}]^*$ of SPT for multivariate integration was estimated in [4], and we have $[\tau^{\text{int}}]^* \in [B, 2B]$. From Theorem 1 it follows that

$$[\tau^{\text{int}}]^* \leq [\tau^{\text{app}}]^* \leq B + \frac{\log 3}{\alpha^*},$$

where $[\tau^{\text{app}}]^*$ is the exponent of SPT for the approximation problem. Hence we have

$$[\tau^{\text{int}}]^* \in \left[B, B + \min \left(B, \frac{\log 3}{\alpha^*} \right) \right],$$

which is an improvement of the result from [4] whenever $\alpha^* > (\log 3)/B$ which means that $a_j > \exp(j(\alpha^* - \delta))$ for large j . If $\alpha^* = \infty$ then

$$[\tau^{\text{int}}]^* = [\tau^{\text{app}}]^* = B.$$

This is the case when $a_j \geq (1 + \alpha)^{b_j}$ for large j and $\alpha > 0$.

- Weak tractability for the integration problem was considered in [4] with a more demanding notion of WT. Suppose that we relax the notion of WT from [4], and use the notion of WT studied in this paper. That is, we say that the integration problem is weakly tractable if

$$\lim_{s + \log \varepsilon^{-1} \rightarrow \infty} \frac{\log n^{\text{int}}(\varepsilon, s)}{s + \log \varepsilon^{-1}} = 0. \quad (28)$$

We stress that the notion of WT as discussed in [4] implies (28), but this does not hold the other way round.

Using the definition (28), we now show that we have the same condition $\lim_j a_j = \infty$ for WT for the integration and approximation problems. Indeed, by Theorem 1, the condition $\lim_j a_j = \infty$ implies WT for the approximation problem, which, by (27), also implies WT for the integration problem. To show the converse, assume that the a_j 's are bounded, say $a_j \leq A < \infty$ for all $j \in \mathbb{N}$. From [4, Corollary 1] it follows that for all $n < 2^s$ we have

$$e^{\text{int}}(n, s) \geq 2^{-s/2} \omega^{2^{-1} \sum_{j=1}^s a_j} \geq 2^{-s/2} \omega^{As/2} = \eta^s,$$

where $\eta := (\omega^A/2)^{1/2} \in (0, 1)$. Hence, for $\varepsilon = \eta^s/2$ we have $e^{\text{int}}(n, s) > \varepsilon$ for all $n < 2^s$. This implies that $n^{\text{int}}(\varepsilon, s) \geq 2^s$ and

$$\frac{\log n^{\text{int}}(\varepsilon, s)}{s + \log \varepsilon^{-1}} \geq \frac{s \log 2}{s + \log 2 + s \log \eta^{-1}} \rightarrow \frac{\log 2}{1 + \log \eta^{-1}} > 0 \quad \text{as } s \rightarrow \infty.$$

Thus we do not have WT.

This means that WT holds in the sense of (28) for the integration problem iff $\lim_j a_j = \infty$, which is the same condition as for the approximation problem.

Since for the integration problem we have UEXP iff $B < \infty$, see [4, Theorem 1], it follows that we have WT+UEXP iff $B < \infty$ and $\lim_j a_j = \infty$. Again, this is the same condition as for the approximation problem.

10 Analyticity of functions from $H(K_{s,a,b})$

In this section we show that the functions from the Korobov space $H(K_{s,a,b})$ are analytic.

Proposition 2 *Functions $f \in H(K_{s,a,b})$ are analytic.*

Proof. Since $H(K_{s,a,b}) \subseteq H(K_{s,1,1})$ it suffices to show the assertion for $f \in H(K_{s,1,1})$.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{N}_0^s$ with $|\alpha| = \alpha_1 + \dots + \alpha_s$. For $f \in H(K_{s,1,1})$, consider the operator D^α of partial differentiation,

$$D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_s^{\alpha_s}} f.$$

Then

$$D^\alpha f(\mathbf{x}) = \sum_{\mathbf{h} \in \mathbb{Z}^s} \left[\widehat{f}(\mathbf{h}) (2\pi \mathbf{i})^{|\alpha|} \prod_{j=1}^s h_j^{\alpha_j} \right] \exp(2\pi \mathbf{i} \mathbf{h} \cdot \mathbf{x}),$$

where, by convention, we take $0^0 = 1$.

Let $\omega_1 \in (\omega, 1)$ and $q = \omega/\omega_1 < 1$. For any $\alpha \in \mathbb{N}$ consider $g(x) = x^{2\alpha} q^x$ for $x \geq 0$. Then $g'(x) = 0$ if $x = 2\alpha/\log q^{-1}$ and

$$g''(2\alpha/\log q^{-1}) = \frac{1}{2} \left(\frac{2}{e} \right)^{2\alpha} \left(\frac{\alpha}{\log q^{-1}} \right)^{2\alpha-1} \log q < 0.$$

Hence,

$$g(x) \leq g(2\alpha/\log q^{-1}) = \left(\frac{2\alpha}{e \log q^{-1}}\right)^{2\alpha}.$$

Since

$$\alpha^{2\alpha} = \left(\alpha! \frac{\alpha^\alpha}{\alpha!}\right)^2 \leq e^{2\alpha} (\alpha!)^2$$

then

$$g(x) \leq \left(\frac{2}{\log q^{-1}}\right)^{2\alpha} (\alpha!)^2.$$

Hence, we have

$$x^{2\alpha} \omega^x \leq \left(\frac{2}{\log \omega_1 - \log \omega}\right)^{2\alpha} (\alpha!)^2 \omega_1^x =: C^{2\alpha} (\alpha!)^2 \omega_1^x.$$

Note that C depends only on ω and ω_1 .

Then $\omega_{\mathbf{h}} = \omega^{|h_1|+\dots+|h_s|}$ implies

$$\begin{aligned} |D^\alpha f(\mathbf{x})| &= \left| \sum_{\mathbf{h} \in \mathbb{Z}^s} [\widehat{f}(\mathbf{h}) \omega_{\mathbf{h}}^{-1/2}] \left[\omega_{\mathbf{h}}^{1/2} (2\pi \mathbf{i})^{|\alpha|} \prod_{j=1}^s h_j^{\alpha_j} \right] \exp(2\pi \mathbf{i} \mathbf{h} \cdot \mathbf{x}) \right| \\ &\leq \|f\|_{H(K_{s,1,1})} \left[\sum_{\mathbf{h} \in \mathbb{Z}^s} (2\pi)^{2|\alpha|} \prod_{j=1}^s |h_j|^{2\alpha_j} \omega^{|h_j|} \right]^{1/2} \\ &\leq \|f\|_{H(K_{s,1,1})} \left[\sum_{\mathbf{h} \in \mathbb{Z}^s} (2\pi)^{2|\alpha|} \prod_{j=1}^s [C^{2\alpha_j} (\alpha_j!)^2] \omega_1^{|h_j|} \right]^{1/2} \\ &\leq \|f\|_{H(K_{s,1,1})} \prod_{j=1}^s [(2\pi C)^{\alpha_j} \alpha_j!] \left[\sum_{\mathbf{h} \in \mathbb{Z}^s} \prod_{j=1}^s \omega_1^{|h_j|} \right]^{1/2} \\ &\leq \|f\|_{H(K_{s,1,1})} (2\pi C)^{|\alpha|} \prod_{j=1}^s (\alpha_j!) \left(1 + \frac{2}{1 - \omega_1}\right)^{s/2} \\ &=: C_1 \cdot C_2^{|\alpha|} \prod_{j=1}^s (\alpha_j!), \end{aligned}$$

where $C_1 = \|f\|_{H(K_{s,1,1})} \left(1 + \frac{2}{1 - \omega_1}\right)^{s/2} \geq 0$ and $C_2 = 2\pi C > 0$.

Then for any $\boldsymbol{\zeta} = (\zeta_1, \dots, \zeta_s)$ and any $\mathbf{x} = (x_1, \dots, x_s)$ with $\|\mathbf{x} - \boldsymbol{\zeta}\|_\infty < C_2^{-1}$ we have

$$\begin{aligned} \left| \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^s} \frac{D^\alpha f(\boldsymbol{\zeta})}{(\alpha_1!) \cdots (\alpha_s!)} \prod_{j=1}^s (x_j - \zeta_j)^{\alpha_j} \right| &\leq C_1 \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^s} \prod_{j=1}^s (C_2 |x_j - \zeta_j|)^{\alpha_j} \\ &\leq C_1 \left(\sum_{\alpha=0}^{\infty} (C_2 \|\mathbf{x} - \boldsymbol{\zeta}\|_\infty)^\alpha \right)^s \\ &= C_1 \left(\frac{1}{1 - C_2 \|\mathbf{x} - \boldsymbol{\zeta}\|_\infty} \right)^s < \infty. \end{aligned}$$

Hence f is analytic, as claimed. \square

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